Structure-preserving finite-element approximations of the magnetic Euler–Poisson equations

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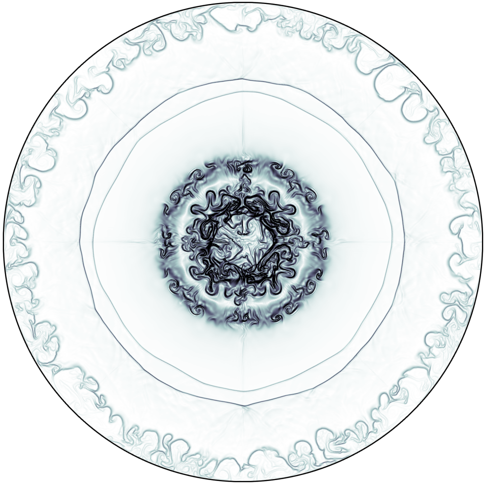
Supported by

- NSF DMS-2045636
- AFOSR FA9550-23-1-0007
- TAMU NLO Research Seed Funding 2025



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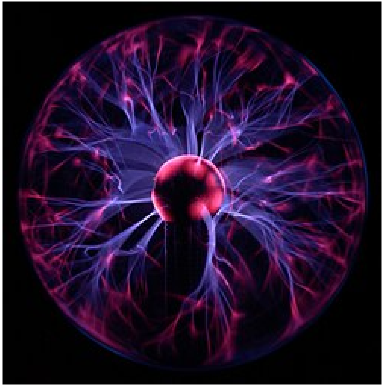
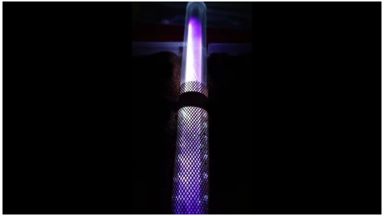
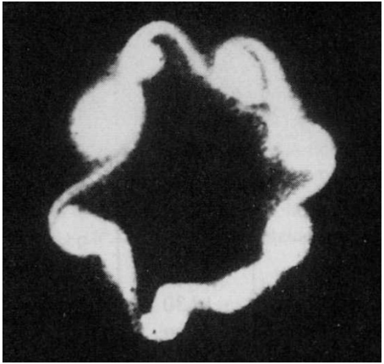
Motivation

Cold plasma physics:







The magnetic Euler–Poisson equations

$$
\begin{cases}
\partial_t \begin{pmatrix} \rho \\ \boldsymbol{m} \\ E \end{pmatrix} + \operatorname{div} \begin{pmatrix} \boldsymbol{m}^\top \\ \rho^{-1}\boldsymbol{m}\boldsymbol{m}^\top + \mathbf{I}p \\ \rho^{-1}\boldsymbol{m}^\top(E+p) \end{pmatrix} = \begin{pmatrix} 0 \\ -\rho\nabla\varphi + \boldsymbol{m}\times\boldsymbol{\Omega} \\ -\nabla\varphi\cdot\boldsymbol{m} \end{pmatrix} \\
\qquad\qquad\qquad\qquad \partial_t(-\Delta\varphi) = -\alpha \operatorname{div} \boldsymbol{m}
\end{cases}
$$

Electron fluid

$$\alpha := \frac{1}{\varepsilon_0}\frac{q_e^2}{m_e^2}$$

charge q_e, mass m_e

Magnetic field

$$\boldsymbol{\Omega} := \frac{q_\mathrm{e}}{m_\mathrm{e}}\boldsymbol{B}$$

magnetic flux density $\boldsymbol{B}$

The magnetic Euler–Poisson equations

What structure is in this system? (pt 1)

Lemma[1]

Consider

$$\partial_t \boldsymbol{u} + \operatorname{div} \mathbf{f}(\boldsymbol{u}) = \boldsymbol{s}(\boldsymbol{g}), \quad \boldsymbol{s}(\boldsymbol{g}) := \begin{pmatrix} 0 \\ \boldsymbol{g} \\ \frac{1}{\rho}\boldsymbol{m} \cdot \boldsymbol{g} \end{pmatrix}.$$

Then, for all Lipschitz $\Psi(\boldsymbol{u}) = \psi\big(\rho, e(\boldsymbol{u})\big)$,

$$\nabla_{\boldsymbol{u}}\Psi(\boldsymbol{u}) \cdot \boldsymbol{s}(\boldsymbol{g}) = 0 \quad \text{a. e.}$$

Corollary[1]

Let $\boldsymbol{u}$ be of class C^1, then

$$\partial_t \Psi(\boldsymbol{u}) + \nabla_{\boldsymbol{u}}\Psi(\boldsymbol{u}) \cdot \operatorname{div} \mathbf{f}(\boldsymbol{u}) = \mathbf{0}.$$

$\longrightarrow$ External forces do not modify the internal energy and entropies.

[1] Maier, Shadid, Tomas '23

The magnetic Euler–Poisson equations

What structure is in this system? (pt 2)

Hyperbolic subsystem:

$$\partial_t \underbrace{\begin{pmatrix} \rho \\ \boldsymbol{m} \\ E \end{pmatrix}}_{\mathbf{u}} + \operatorname{div} \underbrace{\begin{pmatrix} \boldsymbol{m}^\top \\ \rho^{-1}\boldsymbol{m}\boldsymbol{m}^\top + \mathbb{I}p \\ \rho^{-1}\boldsymbol{m}^\top(E+p) \end{pmatrix}}_{\mathbf{f}(\boldsymbol{u})} = \mathbf{0}$$

Invariant domain:

$$\mathcal{A} := \left\{ (\rho, \boldsymbol{m}, E) : \rho > 0,\ \underbrace{E - \frac{1}{2\rho}|\boldsymbol{m}|^2}_{\rho e} > 0,\ s(\rho, e) \geq s_0 \right\}$$

The magnetic Euler–Poisson equations

What structure is in this system? (pt 3)

Source dominated subsystem:

$$\partial_t \begin{pmatrix} \rho \\ \boldsymbol{m} \\ E \end{pmatrix} = \begin{pmatrix} 0 \\ -\rho\nabla\varphi + \boldsymbol{m}\times\boldsymbol{\Omega} \\ -\nabla\varphi\cdot\boldsymbol{m} \end{pmatrix}, \quad \partial_t(-\Delta\varphi) = -\alpha\,\mathrm{div}\boldsymbol{m}.$$

High-frequency oscillations:

Plasma frequency

Assuming $\boldsymbol{\Omega} \approx \mathbf{0}$ and $\nabla\rho \approx 0$:

$$-\Delta\underbrace{\left(\partial_{tt}\varphi + \omega_{\mathrm{p}}^2\varphi\right)}_{\text{harmonic oscillator}} = 0, \quad \omega_{\mathrm{p}} = \sqrt{\rho\alpha}.$$

Cyclotron frequency

Assuming $\nabla\varphi \approx 0$ and $\boldsymbol{m} \perp \boldsymbol{\Omega}$:

$$\underbrace{\partial_{tt}\boldsymbol{m} + \omega_{\mathrm{c}}^2\boldsymbol{m}}_{\text{harmonic oscillator}} = 0, \quad \omega_{\mathrm{c}} = |\boldsymbol{\Omega}|.$$

The magnetic Euler–Poisson equations

What structure is in this system? (pt 4)

But these fast dynamics might be hidden...

Magnetic drift limit

Balanced Lorentz force implies

$$\boldsymbol{v}_{\mathrm{dr}} = -\frac{\nabla\varphi \times \boldsymbol{\Omega}}{|\boldsymbol{\Omega}|^2}.$$

$$\begin{aligned} \partial_t \rho + \operatorname{div}\left(\rho \boldsymbol{v}_{\mathrm{dr}}\right) &= 0, \\ \partial_t(-\Delta\varphi) + \alpha \operatorname{div}\left(\rho \boldsymbol{v}_{\mathrm{dr}}\right) &= 0. \end{aligned}$$

Diocotron frequency

Time scale via dimensional analysis:

$$\omega_{\mathrm{d}} = \frac{\rho\alpha}{|\boldsymbol{\Omega}|} = \frac{\omega_{\mathrm{p}}^2}{\omega_{\mathrm{c}}}.$$

Huge scale separation:

$$\omega_{\mathrm{d}} \sim 1 \ll \omega_{\mathrm{p}} \sim 10^6 \ll \omega_{\mathrm{c}} \sim 10^{12}$$

The magnetic Euler–Poisson equations

What structure is in this system? (pt 5)

Lemma (Formal energy balance)[1]

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_D \left\{E + \frac{1}{2\alpha}\left|\nabla\varphi\right|^2\right\}\mathrm{d}x + \int_{\partial D}\left\{\frac{\boldsymbol{m}}{\rho}(E+p) + \varphi\Big(\boldsymbol{m} - \frac{1}{\alpha}\nabla\partial_t\varphi\Big)\right\}\cdot\boldsymbol{n}\,\mathrm{d}s = 0.$$

Lemma (Gauß law)[1]

If the initial values satisfy the Gauß law, then it is maintained at all times, viz.

$$-\Delta\varphi = \alpha\rho.$$

[1] Maier, Shadid, Tomas '23

The magnetic Euler–Poisson equations

Agenda: use an operator splitting approach

Hyperbolic update

$$\partial_t \boldsymbol{u} + \operatorname{div} \mathbf{f}(\boldsymbol{u}) = \mathbf{0}.$$

Discretize with a scheme that

- is conservative,
- preserves invariant domain $\mathcal{A}$:

$$\Big\{\rho > 0,\ e > 0,\ s \geq \min s_0\Big\}.$$

Source update

$$\partial_t \begin{pmatrix} \rho \\ \boldsymbol{m} \\ E \end{pmatrix} = \begin{pmatrix} 0 \\ -\rho\nabla\varphi + \boldsymbol{m} \times \boldsymbol{\Omega} \\ -\nabla\varphi \cdot \boldsymbol{m} \end{pmatrix},$$

$$\partial_t(-\Delta\varphi) = -\alpha \operatorname{div} \boldsymbol{m}.$$

Discretize with a scheme that

- keeps ρ and e invariant,
- maintains global energy balance,
- does not require to resolve ω_p, ω_c.

The magnetic Euler–Poisson equations

How to discretize?

Source update

$$
\begin{aligned}
\partial_t \boldsymbol{v} &= -\nabla\varphi + \boldsymbol{v} \times \boldsymbol{\Omega}, \\
\partial_t(-\Delta\varphi) &= -\alpha \operatorname{div}(\rho \boldsymbol{v}).
\end{aligned}
$$

Fundamental difficulty:

- $\boldsymbol{u} = [\rho, \boldsymbol{m}, E]^T$ collocated, algebraic quantity
- φ given by weak formulation, control $\nabla\varphi \in L^2$

A PDE Schur complement[1]

Rothe's method: Discretize in time, then discretize in space

Discretize in time with a Crank–Nicolson scheme:

$$\begin{cases} \boldsymbol{v}^{n+1/2} = \boldsymbol{v}^n - \dfrac{\tau_n}{2}\nabla\varphi^{n+1/2} + \dfrac{\tau_n}{2}\boldsymbol{v}^{n+1/2}\times\boldsymbol{\Omega}, & \boldsymbol{v}^{n+1} = 2\boldsymbol{v}^{n+1/2} - \boldsymbol{v}^n, \\ -\Delta\varphi^{n+1/2} = -\Delta\varphi^n - \dfrac{\tau_n}{2}\alpha\,\nabla\cdot\left(\rho^n\boldsymbol{v}^{n+1/2}\right), & \varphi^{n+1} = 2\varphi^{n+1/2} - \varphi^n. \end{cases}$$

Lemma (Energy balance)

Test with $\rho^n\boldsymbol{v}^{n+1/2}$ and with $\frac{1}{\alpha}\varphi^{n+1/2}$ and integrate:

$$\int_D \frac{1}{2}\rho^n|\boldsymbol{v}^{n+1}|^2 + \frac{1}{2\alpha}|\nabla\varphi^{n+1}|^2\,dx = \int_D \frac{1}{2}\rho^n|\boldsymbol{v}^n|^2 + \frac{1}{2\alpha}|\nabla\varphi^n|^2\,dx.$$

[1] Maier, Shadid, Tomas '23

A PDE Schur complement

Schur complement

$$
\begin{cases}
\boldsymbol{v}^{n+1/2} = \mathcal{B}_n^{-1}\left(\boldsymbol{v}^n - \dfrac{\tau_n}{2}\nabla\varphi^{n+1/2}\right), \\
-\Delta\varphi^{n+1/2} - \dfrac{\tau_n^2}{4}\alpha\nabla\cdot\left(\rho^n\mathcal{B}_n^{-1}\nabla\varphi^{n+1/2}\right) = -\Delta\varphi^n - \dfrac{\tau_n}{2}\alpha\nabla\cdot\left(\rho^n\mathcal{B}_n^{-1}\boldsymbol{v}^n\right),
\end{cases}
$$

where

$$
\mathcal{B}_n\boldsymbol{v} = \boldsymbol{v} - \frac{\tau_n}{2}\boldsymbol{v}\times\boldsymbol{\Omega}.
$$

Challenge: discretize in space while maintaining energy balance.

A PDE Schur complement

Given (possibly non-affine) quadrilateral(hexahedral) mesh,

- FE space $\mathbb{H}_h$ — continuous bi(tri)-linear finite elements,
- FE space $\mathbb{V}_h$ — discontinuous bi(tri)-linear finite elements,
- Lumped inner product on $\mathbb{V}_h$:

$$\langle f, g \rangle_h := \sum_K \sum_i m_i^K \, f|_K(\boldsymbol{x}_i) \, g|_K(\boldsymbol{x}_i), \qquad m_i^K = \int_K \varphi_{K,i}^h \mathrm{d}x,$$

where $\boldsymbol{x}_i$ is ith vertex and $\varphi_{K,i}^h$ is the corresponding shape function of $\mathbb{V}_h$ defined on K.

A PDE Schur complement

Fully discrete source update

Given $\varphi_h^n \in \mathbb{H}_h$ and $\boldsymbol{v}_h^n \in \mathbb{V}_h^d$, solve for $\varphi_h^{n+1} \in \mathbb{H}_h$ and $\boldsymbol{v}_h^{n+1} \in \mathbb{V}_h^d$ satisfying

$$\begin{cases} a_h^n(\varphi_h^{n+1/2}, w_h) = (\nabla\varphi_h^n, \nabla w_h)_{L^2(D)} + \dfrac{\tau_n}{2}\alpha\big\langle \rho_h^n \mathcal{B}_n^{-1}\boldsymbol{v}_n^h, \nabla w_h\big\rangle_h, & \forall w_h \in \mathbb{H}_h, \\ \big\langle \boldsymbol{v}_h^{n+1/2}, \boldsymbol{z}_h\big\rangle_h = \big\langle \mathcal{B}_n^{-1}(\boldsymbol{v}_h^n - \dfrac{\tau_n}{2}\nabla\varphi_h^{n+1/2}), \boldsymbol{z}_h\big\rangle_h, & \forall \boldsymbol{z}_h \in \mathbb{V}_h^d, \\ \varphi_h^{n+1} = 2\varphi_h^{n+1/2} - \varphi_h^n, & \\ \boldsymbol{v}_h^{n+1} = 2\boldsymbol{v}_h^{n+1/2} - \boldsymbol{v}_h^n. & \end{cases}$$

Lemma

$$a_h^n(\varphi_h, w_h) := (\nabla\varphi_h, \nabla w_h)_{L^2(D)} + \frac{\tau_n^2}{4}\alpha\left\langle \rho_h^n \mathcal{B}_n^{-1}\nabla\varphi_h, \nabla w_h\right\rangle_h$$

is bounded and coercive on $\mathbb{H}_h$ with coefficients independent of $\omega_c = |\boldsymbol{\Omega}|$ and h:

$$\|\nabla\varphi_h\|_{L^2(D)}^2 \le a_h^n(\varphi_h, \varphi_h), \qquad a_h^n(\varphi_h, w_h) \le C\big(1 + \mathcal{O}(\tau_n^2\omega_{\mathrm{p}}^2)\big)\|\varphi_h\|_{L^2(D)}\|w_h\|_{L^2(D)}.$$

A PDE Schur complement

Theorem (Discrete energy stability)[1]

The fully discrete source update admits an energy balance:

$$\frac{1}{2}\left\langle \rho_h^n \boldsymbol{v}_h^{n+1}, \boldsymbol{v}_h^{n+1} \right\rangle_h + \frac{1}{2\alpha}\|\nabla \varphi_h^{n+1}\|^2_{\boldsymbol{L}^2(D)} = \frac{1}{2}\left\langle \rho_h^n \boldsymbol{v}_h^n, \boldsymbol{v}_h^n \right\rangle_h + \frac{1}{2\alpha}\|\nabla \varphi_h^n\|^2_{\boldsymbol{L}^2(D)}.$$

Corollary (Hyperbolic energy update)[1]

Setting

$$E_i^{n+1} = E_i^n + \frac{1}{2}\rho_i^n\left(|\boldsymbol{v}_i^{n+1}|^2 - |\boldsymbol{v}_i^n|^2\right)$$

maintains global energy balance and keeps ρ and e invariant.

A PDE Schur complement

Remark:

Discrete Gauss law violation

The source update does not guarantee

$$\int_D \nabla \varphi_h^{n+1} \cdot \nabla w_h \,\mathrm{d}x = \alpha \left\langle \rho_h^{n+1}, w_h \right\rangle_h$$

for $w_h \in \mathbb{H}_h$.

No restart

Accept φ_h^{n+1} as is.

$\longrightarrow$ No Gauss law...

Full restart (at end of step)

Recompute φ_h^{n+1}.

$\longrightarrow$ No energy balance...

A PDE Schur complement

Remark:

Relaxation

Compute $\tilde{\varphi}_h^{n+1}$ with Gauß law, with

$$\begin{aligned} \delta E &:= \frac{1}{2\alpha}\left(\|\nabla\tilde{\varphi}_h^{n+1}\|^2_{L^2(D)} - \|\nabla\varphi_h^{n+1}\|^2_{L^2(D)}\right) \\ \mathrm{kin} &:= \frac{1}{2}\left\langle \rho_h^n \boldsymbol{v}_h^{n+1}, \boldsymbol{v}_h^{n+1}\right\rangle_h \end{aligned}$$

update

$$\boldsymbol{m}_i^{n+1} \leftarrow \sqrt{1 - \left(\frac{\delta E}{\mathrm{kin}}\right)_+}\,\boldsymbol{m}_i^{n+1}.$$

$\longrightarrow$ Reestablish energy balance by lowering kinetic energy.

Computational results

Diocotron instability for low-density plasma

$$\omega_{\mathrm{d}} \sim 1 \ll \omega_{\mathrm{p}} \sim 10^6 \ll \omega_{\mathrm{c}} \sim 10^{12}$$

- Isothermal equation of state,

$$p = c^2 \rho.$$

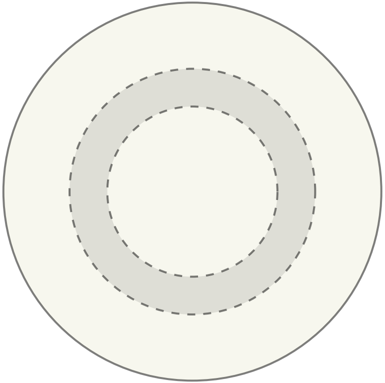
- Initial conditions:

$$\rho_0(\boldsymbol{x}) := \begin{cases} 1, & \text{inside annulus,} \\ 10^{-6}, & \text{outside annulus.} \end{cases}$$

$$\boldsymbol{v}_0(\boldsymbol{x}) := \boldsymbol{v}_{\mathrm{dr}}(\boldsymbol{x}) = -\frac{\nabla\varphi(\boldsymbol{x}) \times \boldsymbol{\Omega}}{|\boldsymbol{\Omega}|^2}.$$

- Linear stability: (nth mode)[1]

$$\text{growth} \sim \{\text{geom. \& n}\}\,\omega_d.$$

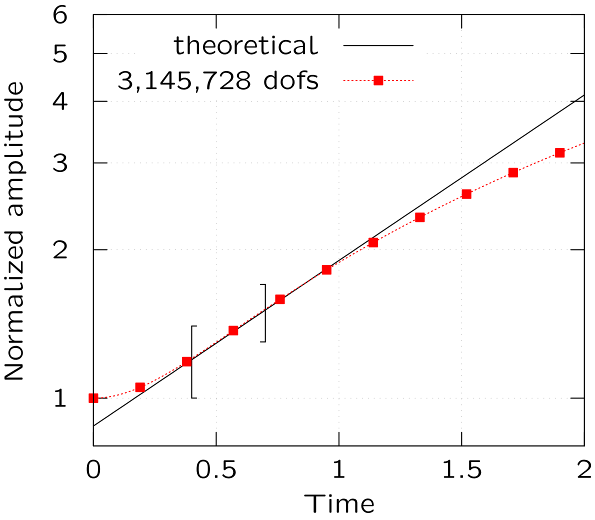


(3rd mode) (5th mode)

[1] Davidson & Felice ’98

Computational results

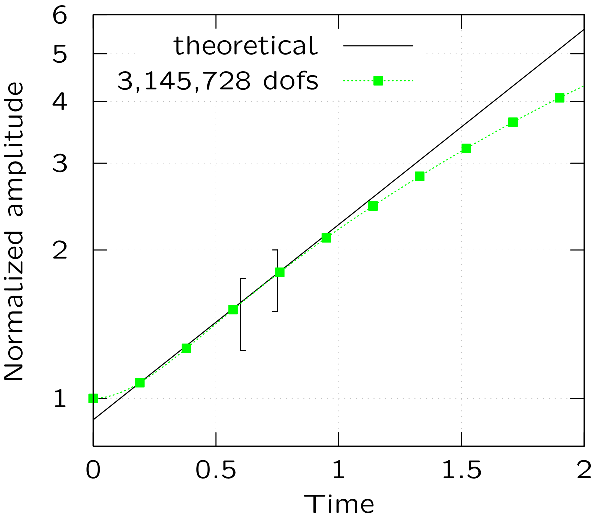
Diocotron instability growth rates



ℓ	dofs	$\gamma_{\ell,h}$	deviation
	196,608	0.777	0.005
3	786,432	0.775	0.003
	3,145,728	0.773	0.001

Computational results

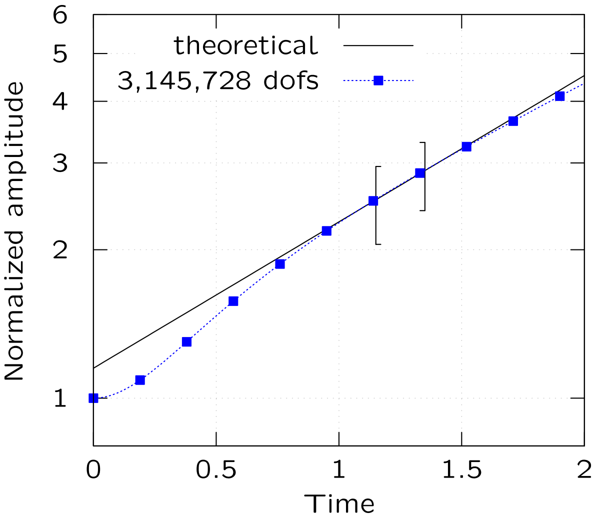
Diocotron instability growth rates



ℓ	dofs	$\gamma_{\ell,h}$	deviation
	196,608	0.935	0.024
4	786,432	0.919	0.008
	3,145,728	0.913	0.002

Computational results

Diocotron instability growth rates



ℓ	dofs	$\gamma_{\ell,h}$	deviation
	196,608	0.667	0.016
5	786,432	0.677	0.006
	3,145,728	0.680	0.003

Conclusion and Outlook

- We considered the magnetic Euler–Poisson equations where the magnetic field is known but the electric potential is unknown
- We developed an efficient numerical method that preserves the relevant structure of the PDE
- Future work: Euler–Maxwell equations

`github.com/conservation-laws/ryujin`

`dealii.org`

"*Structure-preserving finite-element approximations of the magnetic Euler–Poisson equations*", J. Hoffart, M. Maier, J. Shadid, and I. Tomas, arXiv preprint (2025)

Thank you for your attention!