# Towards an involution-preserving solver for the time-dependent Maxwell equations

Jordan Hoffart

Department of Mathematics
Texas A&M University

# Acknowledgments


## Funding

National Science Foundation (DMS-2045636)

Air Force Office of Scientific Research (FA9550-23-1-0007)

## Collaborators

Alexandre Ern (CERMICS, Ecole des Ponts, and INRIA)

Jean-Luc Guermond (Texas A&M University)

Matthias Maier (Texas A&M University)

# Introduction

## Long-term goal: Multiphysics systems

- Invariant-domain preserving
- Involution preserving

## Example: Euler-Maxwell

- Positivity of density
- Positivity of internal energy
- Gauss's laws

## Today's goal: Maxwell eigenvalue problem

- Involution preserving
- Spectrally correct

# Eigenvalue problem

## Setting

$D \subset \mathbb{R}^d$ polyhedral, $d = 3$

$\boldsymbol{H}(\mathbf{curl}, D) = \{\boldsymbol{v} \in \boldsymbol{L}^2(D) : \nabla \times \boldsymbol{v} \in \boldsymbol{L}^2(D)\}$

$\boldsymbol{H}_0(\mathbf{curl}, D) = \{\boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl}, D) : \boldsymbol{v} \times \boldsymbol{n} = \mathbf{0} \text{ on } \partial D\}$

## Problem

Find $\lambda \in \mathbb{C} \setminus \{0\}$ and $(\boldsymbol{H}, \boldsymbol{E}) \in \boldsymbol{H}_0(\mathbf{curl}, D) \times \boldsymbol{H}(\mathbf{curl}, D)$ such that

$$-\nabla \times \boldsymbol{E} = \frac{1}{\lambda}\boldsymbol{H} \qquad \nabla_0 \times \boldsymbol{H} = \frac{1}{\lambda}\boldsymbol{E}$$

## Involutions

$\boldsymbol{H} \in \mathrm{im}(\nabla\times)$ and $\boldsymbol{E} \in \mathrm{im}(\nabla_0\times)$

# Involutions

## Orthogonality conditions

$$\boldsymbol{H} \in \mathrm{im}(\nabla\times) \iff \boldsymbol{H} \perp \ker(\nabla_0\times) := \boldsymbol{H}_0(\mathbf{curl} = \mathbf{0}, D)$$

$$\boldsymbol{E} \in \mathrm{im}(\nabla_0\times) \iff \boldsymbol{E} \perp \ker(\nabla\times) := \boldsymbol{H}(\mathbf{curl} = \mathbf{0}, D)$$

## Orthogonal projections

$$\boldsymbol{\Pi}_0^c : \boldsymbol{L}^2(D) \to \boldsymbol{H}_0(\mathbf{curl} = \mathbf{0}, D)$$

$$\boldsymbol{\Pi}^c : \boldsymbol{L}^2(D) \to \boldsymbol{H}(\mathbf{curl} = \mathbf{0}, D)$$

## Involution-preserving subspaces

$$\boldsymbol{X}_0^c = \{\boldsymbol{H} \in \boldsymbol{H}_0(\mathbf{curl}, D) : \boldsymbol{\Pi}_0^c \boldsymbol{H} = \mathbf{0}\} \qquad \boldsymbol{X}^c = \{\boldsymbol{E} \in \boldsymbol{H}(\mathbf{curl}, D) : \boldsymbol{\Pi}^c \boldsymbol{E} = \mathbf{0}\}$$

# Well-posedness

## Boundary-value problem

Given $(\boldsymbol{f}, \boldsymbol{g}) \in L^c(D) := \boldsymbol{L}^2(D) \times \boldsymbol{L}^2(D)$, find $(\boldsymbol{H}, \boldsymbol{E}) \in \boldsymbol{X}_0^c \times \boldsymbol{X}^c$ such that

$$-\nabla \times \boldsymbol{E} = (\boldsymbol{I} - \boldsymbol{\Pi}_0^c)\boldsymbol{f} \qquad \nabla_0 \times \boldsymbol{H} = (\boldsymbol{I} - \boldsymbol{\Pi}^c)\boldsymbol{g}$$

## Lemma (Ern and Guermond)

BVP well-posed with compact solution operator $T : L^c(D) \to L^c(D)$

$(\lambda, (\boldsymbol{H}, \boldsymbol{E}))$ eigenpair of $T$ iff $(\lambda, (\boldsymbol{H}, \boldsymbol{E}))$ solves Maxwell EVP

The eigenvalue problem is well-posed

# Discrete setting

**Mesh family** $\{\mathcal{T}_h\}_h$ shape-regular and affine simplicial

**Polynomial spaces**

$\mathbb{P}_{k,d}$ $d$-variable polynomials total degree at most $k$

$\boldsymbol{P}_k^b(\mathcal{T}_h) = \{\boldsymbol{p} \in \boldsymbol{L}^2(D) : \boldsymbol{p}|_K \in (\mathbb{P}_{k,d})^d \text{ for all } K \in \mathcal{T}_h\}$

**Jumps and averages**

$\mathcal{F}_h$ mesh faces composed of interfaces $\mathcal{F}_h^\circ$ and boundary faces $\mathcal{F}_h^\partial$

Interfaces $F \in \mathcal{F}_h^\circ$ have a left cell $K_{l,F}$ and a right cell $K_{r,F}$

Tangential jump $[\![\boldsymbol{p}_h]\!]_F^c = (\boldsymbol{p}_h|_{K_{l,F}} - \boldsymbol{p}_h|_{K_{r,F}}) \times \boldsymbol{n}_{K_{l,F}}$

Average $\{\!\{\boldsymbol{p}_h\}\!\}_F^g = (\boldsymbol{p}_h|_{K_{l,F}} + \boldsymbol{p}_h|_{K_{r,F}})/2$

# Discretization

## Discrete curls

$$(\boldsymbol{C}_{h0}\boldsymbol{H}_h, \boldsymbol{e}_h)_{\boldsymbol{L}^2(D)} = (\nabla_h \times \boldsymbol{H}_h, \boldsymbol{e}_h)_{\boldsymbol{L}^2(D)} + \sum_{F\in\mathcal{F}_h} ([\![\boldsymbol{H}_h]\!]_F^c, \{\!\{\boldsymbol{e}_h\}\!\}_F^g)_{\boldsymbol{L}^2(F)}$$

$$(\boldsymbol{C}_h\boldsymbol{E}_h, \boldsymbol{h}_h)_{\boldsymbol{L}^2(D)} = (\nabla_h \times \boldsymbol{E}_h, \boldsymbol{h}_h)_{\boldsymbol{L}^2(D)} + \sum_{F\in\mathcal{F}_h^\circ} ([\![\boldsymbol{E}_h]\!]_F^c, \{\!\{\boldsymbol{h}_h\}\!\}_F^g)_{\boldsymbol{L}^2(F)}$$

## Stabilization

$$s_h^c(\boldsymbol{H}_h, \boldsymbol{h}_h) = \sum_{F\in\mathcal{F}_h} ([\![\boldsymbol{H}_h]\!]_F^c, [\![\boldsymbol{h}_h]\!]_F^c)_{\boldsymbol{L}^2(F)}$$

$$s_h^{c,\circ}(\boldsymbol{E}_h, \boldsymbol{e}_h) = \sum_{F\in\mathcal{F}_h^\circ} ([\![\boldsymbol{E}_h]\!]_F^c, [\![\boldsymbol{e}_h]\!]_F^c)_{\boldsymbol{L}^2(F)}$$

# Discrete problem

## Discontinuous Galerkin bilinear form

$$a_h((\boldsymbol{H}_h, \boldsymbol{E}_h), (\boldsymbol{h}_h, \boldsymbol{e}_h)) = (\boldsymbol{C}_{h0}\boldsymbol{H}_h, \boldsymbol{e}_h) - (\boldsymbol{C}_h\boldsymbol{E}_h, \boldsymbol{h}_h) + s_h^c(\boldsymbol{H}_h, \boldsymbol{h}_h) + s_h^{c,\circ}(\boldsymbol{E}_h, \boldsymbol{e}_h)$$

## Discrete eigenvalue problem

Find $\lambda_h \neq 0$ and $(\boldsymbol{H}_h, \boldsymbol{E}_h) \in L_h^c := \boldsymbol{P}_k^b(\mathcal{T}_h) \times \boldsymbol{P}_k^b(\mathcal{T}_h)$ such that

$$a_h((\boldsymbol{H}_h, \boldsymbol{E}_h), (\boldsymbol{h}_h, \boldsymbol{e}_h)) = \frac{1}{\lambda_h}((\boldsymbol{H}_h, \boldsymbol{h}_h)_{\boldsymbol{L}^2(D)} + (\boldsymbol{E}_h, \boldsymbol{e}_h)_{\boldsymbol{L}^2(D)})$$

for all $(\boldsymbol{h}_h, \boldsymbol{e}_h) \in L_h^c$.

# Discrete involutions

## Discrete operator

$$(A_h(\boldsymbol{H}_h, \boldsymbol{E}_h), (\boldsymbol{h}_h, \boldsymbol{e}_h))_{L_h^c} = a_h((\boldsymbol{H}_h, \boldsymbol{E}_h), (\boldsymbol{h}_h, \boldsymbol{e}_h))$$

## Discrete eigenvalue problem

$(\lambda_h, (\boldsymbol{H}_h, \boldsymbol{E}_h))$ solves discrete EVP iff

$$A_h(\boldsymbol{H}_h, \boldsymbol{E}_h) = \frac{1}{\lambda_h}(\boldsymbol{H}_h, \boldsymbol{E}_h)$$

## Discrete involutions

$$(\boldsymbol{H}_h, \boldsymbol{E}_h) \in \operatorname{im} A_h = (\ker(A_h^\top))^\perp$$

# Characterizing the discrete involutions

## Nédélec polynomials

$\boldsymbol{P}_k^c(\mathcal{T}_h)$ curl-conforming piecewise Nédélec polynomials degree $k \geq 0$

$\boldsymbol{P}_{k0}^c(\mathcal{T}_h) = \{\boldsymbol{p} \in \boldsymbol{P}_k^c(\mathcal{T}_h) : \boldsymbol{p} \times \boldsymbol{n} = \boldsymbol{0}\}$

## Lemma (Ern and Guermond)

For affine simplicial meshes, $\ker(A_h^\top) = \boldsymbol{P}_{k0}^c(\mathbf{curl} = \boldsymbol{0}, \mathcal{T}_h) \times \boldsymbol{P}_k^c(\mathbf{curl} = \boldsymbol{0}, \mathcal{T}_h)$.

## Discrete projections

$\boldsymbol{\Pi}_{h0}^c : \boldsymbol{L}^2(D) \to \boldsymbol{P}_{k0}^c(\mathbf{curl} = \boldsymbol{0}, \mathcal{T}_h)$ $\qquad$ $\boldsymbol{\Pi}_h^c : \boldsymbol{L}^2(D) \to \boldsymbol{P}_k^c(\mathbf{curl} = \boldsymbol{0}, \mathcal{T}_h)$

## Discrete involution-preserving subspaces

$\boldsymbol{X}_{h0}^c = \{\boldsymbol{H}_h \in \boldsymbol{P}_k^b(\mathcal{T}_h) : \boldsymbol{\Pi}_{h0}^c \boldsymbol{H}_h = \boldsymbol{0}\}$ $\qquad$ $\boldsymbol{X}_h^c = \{\boldsymbol{E}_h \in \boldsymbol{P}_k^b(\mathcal{T}_h) : \boldsymbol{\Pi}_h^c \boldsymbol{E}_h = \boldsymbol{0}\}$

# Spectral correctness

## Discrete boundary-value problem

Given $(\boldsymbol{f}, \boldsymbol{g}) \in L^c(D)$, find $(\boldsymbol{H}_h, \boldsymbol{E}_h) \in \boldsymbol{X}^c_{h0} \times \boldsymbol{X}^c_h$ such that

$$a_h((\boldsymbol{H}_h, \boldsymbol{E}_h), (\boldsymbol{h}_h, \boldsymbol{e}_h)) = ((\boldsymbol{I} - \boldsymbol{\Pi}^c_{h0})\boldsymbol{f}), \boldsymbol{h}_h)_{\boldsymbol{L}^2(D)} + ((\boldsymbol{I} - \boldsymbol{\Pi}^c_h)\boldsymbol{g}), \boldsymbol{e}_h)_{\boldsymbol{L}^2(D)}$$

for all $(\boldsymbol{h}_h, \boldsymbol{e}_h) \in \boldsymbol{X}^c_{h0} \times \boldsymbol{X}^c_h$.

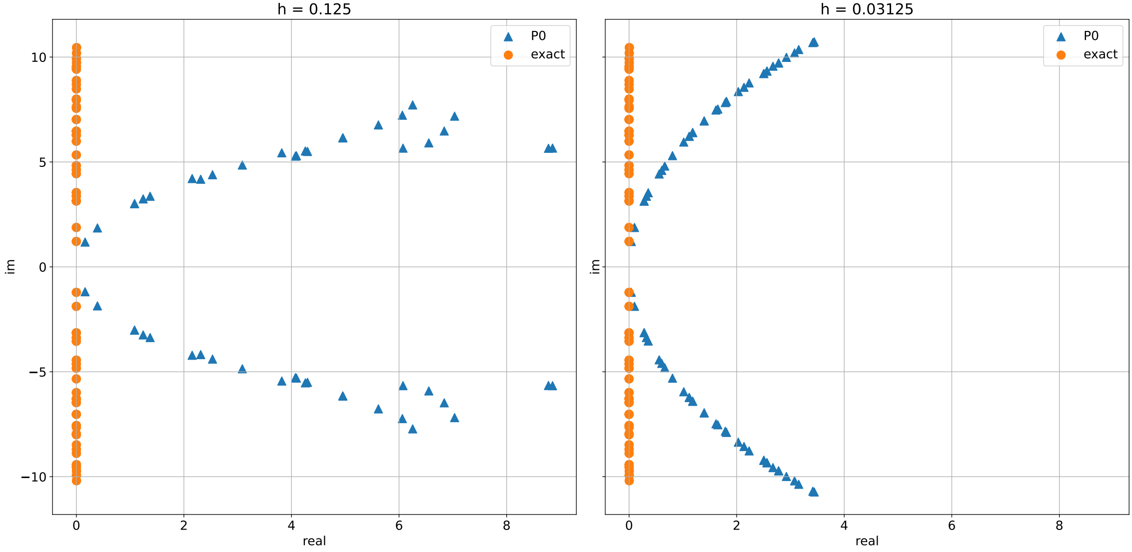
## Theorem (Ern and Guermond)

Discrete BVP well-posed with compact solution operator $T_h : L^c(D) \to L^c(D)$

$(\lambda_h, (\boldsymbol{H}_h, \boldsymbol{E}_h))$ eigenpair $T_h$ iff $(\lambda_h, (\boldsymbol{H}_h, \boldsymbol{E}_h))$ solves discrete EVP

$\|T - T_h\|_{L^c_h} \to 0$ as $h \to 0$ (spectral correctness)

# Spectrally correct simplicial elements



# The hexahedral case

Mesh family $\{\mathcal{T}_h\}_h$ hexahedral

Polynomial spaces

$\mathbb{Q}_{k,d}$ $d$-variable polynomials degree at most $k$ in each variable separately

$\boldsymbol{Q}_k^b(\mathcal{T}_h) = \{\boldsymbol{w}_h \in \boldsymbol{L}^2(D) : \boldsymbol{w}_h|_K \in (\mathbb{Q}_{k,d})^d \text{ for all } K \in \mathcal{T}_h\}$

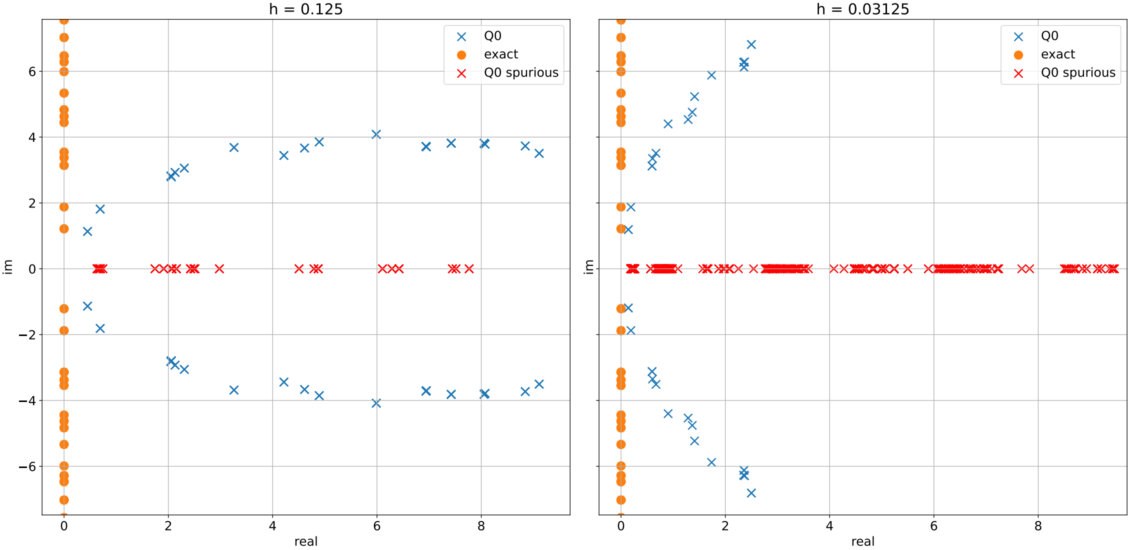
$\boldsymbol{N}_k^c(\mathcal{T}_h)$ curl-conforming piecewise Cartesian Nédélec polynomials

Lemma

For hexahedral meshes,

$$\ker(A_h^\top) \subsetneq \boldsymbol{N}_{k0}^c(\mathbf{curl} = \mathbf{0}, \mathcal{T}_h) \times \boldsymbol{N}_k^c(\mathbf{curl} = \mathbf{0}, \mathcal{T}_h).$$

# Spurious eigenvalues for quadrilateral elements



# Conclusion

The discontinuous Galerkin approximation of the Maxwell eigenvalue problem in first-order form is involution-preserving and spectrally correct on tetrahedral meshes.

Hexahedral meshes still produce spurious modes; future work hopes to fix this.

Next steps: the Euler-Maxwell equations and MHD.

# References

*Spectral correctness of the discontinuous Galerkin approximation of the first-order form of Maxwell's equations with discontinuous coefficients*, Alexandre Ern and Jean-Luc Guermond, 2023

*The discontinuous Galerkin approximation of the Maxwell eigenvalue problem in first-order form on quadrilaterals is spurious*, Jordan Hoffart, 2024

*The deal.II finite element library*, `https://www.dealii.org/`

Thank you!