Completing the Square and the Quadratic Formula

Suppose we are given integers a, b, and c. Suppose also that we are given a complex number x such that

$$ax^2 + bx + c = 0.$$

What is x in terms of a, b, and c? In general, this is not very straightforward to solve. Let's look at an easier, related problem.

Suppose we are given real numbers d, e, f. Suppose some complex number x satisfies

$$(dx+e)^2 + f = 0.$$

What is x in terms of d, e and f? This is much easier to solve:

$$(dx+e)^2 + f = 0 \longrightarrow$$

$$(dx+e)^2 = -f \longrightarrow$$

$$dx+e = \sqrt{-f} \quad \text{or}$$

$$-(dx+e) = \sqrt{-f} \longrightarrow$$

$$dx + e = \pm\sqrt{-f} \longrightarrow$$

$$x = \frac{-e \pm \sqrt{-f}}{d}.$$

Also notice that

$$(dx+e)^2 + f = d^2x^2 + 2dex + e^2 + f$$

is a quadratic polynomial in x.

Let's return to the first problem.

If we can find numbers d, e, and f in terms of a, b, and c such that $ax^2 + bx + C = (dx+e)^2 + f$, then solving $ax^2+bx+c=0$ for x

reduces to solving $(dx+e)^2+f=0$,

which we already know how to do.

Is this possible? What I mean

is, given integers a, b, and c,

can we always find real numbers

d, e, and f such that for all

x,

$$ax^2+bx+c = (dx+e)^2+f \quad ?$$

Let's see: If

$$ax^2 + bx + c = (dx+e)^2 + f,$$

then $ax^2 + bx + c = d^2 x + 2dex + e^2 + f.$

By comparing coefficients, we get a system of equations

$$\begin{cases} a = d^2 \\ b = 2de \\ c = e^2 + f \end{cases}$$

The first equation implies $d = \sqrt{a}$.

The second then implies $e = \dfrac{b}{2\sqrt{a}}$.

The third then implies $f = c - \dfrac{b^2}{4a}$.

We have successfully solved for d, e, and f in terms of a, b, and c. Therefore, since $ax^2+bx+c = (dx+e)^2+f$, substituting for d, e, and f gives us

$$\boxed{ax^2+bx+c = \left(\sqrt{a}\,x + \frac{b}{2\sqrt{a}}\right)^2 + c - \frac{b^2}{4a}}.$$

The form of the quadratic on the right has a special name: it is called <u>completing the square</u>.

Let's look at an example. Let's complete the square on the polynomial

$$3x^2 + 2x + 1.$$

One way to do this is to directly apply the equation we derived previously with $a=3$, $b=2$, and $c=1$.

$$3x^2+2x+1 = \left(\sqrt{3}\,x + \frac{2}{2\sqrt{3}}\right)^2 + 1 - \frac{2^2}{4\cdot 3}$$

If you ever forget the general equation, you can just use the following strategy: We know completing the square means to find d, e, f such that $3x^2+2x+1 = (dx+e)^2+f$.

Expand the right side to get

$$3x^2+2x+1 = d^2x^2+2dex+e^2+f.$$

Compare coefficients to get

$$3 = d^2 \longrightarrow d=\sqrt{3}$$
$$2 = 2de = 2\sqrt{3}e \longrightarrow e = 1/\sqrt{3}$$
$$1 = e^2+f = \frac{1}{3}+f \longrightarrow f = 2/3.$$

Then $3x^2+2x+1 = \left(\sqrt{3}x+\frac{1}{\sqrt{3}}\right)^2 + 2/3.$

Now we know how to complete the square on any polynomial ax^2+bx+c with integer coefficients. Let's see if we can now solve

$$ax^2+bx+c=0 \quad \text{for } x.$$

Completing the square gives us

$$\left(\sqrt{a}\,x+\frac{b}{2\sqrt{a}}\right)^2+c-\frac{b^2}{4a}=0.$$

Then $$\left(\sqrt{a}\,x+\frac{b}{2\sqrt{a}}\right)^2=\frac{b^2}{4a}-c=\frac{b^2-4ac}{4a}$$

Therefore $$\sqrt{a}\,x+\frac{b}{2\sqrt{a}}=\frac{\pm\sqrt{b^2-4ac}}{2\sqrt{a}}.$$

Then $\sqrt{a}\, x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2\sqrt{a}}$.

Finally,

$$\boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}.$$

We have finally solved the problem of finding x such that

$$ax^2 + bx + c = 0.$$

The solution for x above is called the quadratic formula, and it comes from completing the square.

Let's look at another example. Suppose we wish to solve

$$x^2 + x + 1 = 0$$

for x. One way is to directly apply the quadratic formula with $a = b = c = 1$:

$$x = \frac{-1 \pm \sqrt{1^2 - 4(1)(1)}}{2(1)}$$

Another approach, if you ever forget the quadratic formula, is to just complete the square:

$$x^2 + x + 1 = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4}.$$

Then we have that

$$\left(x + \frac{1}{2}\right)^2 + 3/4 = 0 \longrightarrow$$

$$\left(x + 1/2\right)^2 = -3/4 \longrightarrow$$

$$x + 1/2 = \frac{\pm\sqrt{-3}}{2} \longrightarrow$$

$$x = \frac{-1 \pm \sqrt{-3}}{2}.$$

Either approach gives us the same result.