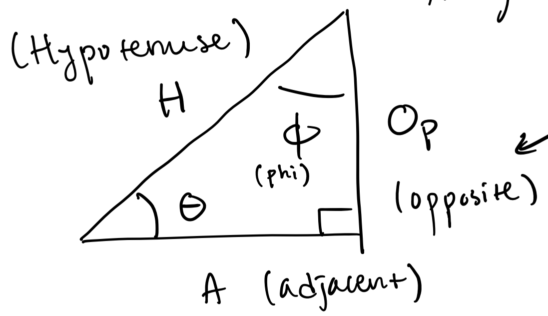
For any <u>acute angle</u> θ (theta) and any right triangle of the form :

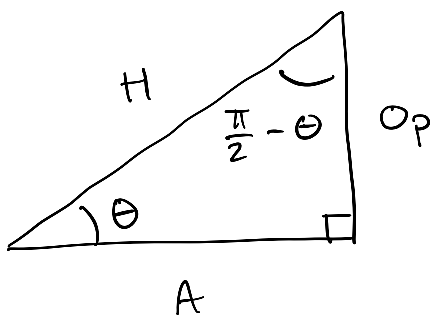


$$\boxed{\begin{array}{lll} \sin\theta \ \text{(sine)} & := & \dfrac{Op}{H} \\[2ex] \tan\theta \ \text{(tangent)} & := & \dfrac{Op}{A} \\[2ex] \sec\theta \ \text{(secant)} & := & \dfrac{H}{A} \end{array}}$$

FACT: The angles of a triangle add up to $180°$ (π radians). Also, $90° = \pi/2$ radians.

Therefore $\theta + \pi/2 + \phi = \pi \longrightarrow$

$\theta + \phi = \pi/2 \longrightarrow \phi = \pi/2 - \theta$.



We call $\frac{\pi}{2} - \theta$ the <u>complementary</u> angle to θ.

(by definition)

(cosine)

Now $\sin(\pi/2 - \theta) = \boxed{\dfrac{A}{H} := \cos(\theta)}$, i.e.

the sine of the complementary angle is defined as the complementary sine (cosine) of the angle.

(by definition)

$$\tan(\pi/2 - \theta) = \boxed{\frac{A}{Op} := \underset{\text{(cotangent)}}{\cot(\theta)}}, \quad \text{ie}$$

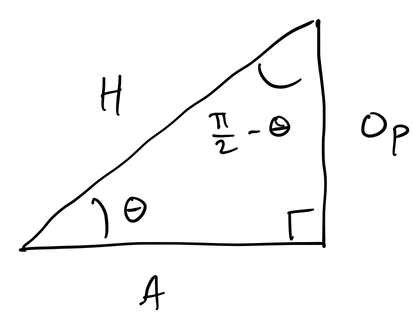
the tangent of the complementary angle is defined to be the complementary tangent (cotangent) of the angle.

(by definition)

$$\sec(\pi/2 - \theta) = \boxed{\frac{H}{Op} := \underset{\text{(cosecant)}}{\csc(\theta)}}, \quad \text{i.e.}$$

the secant of the complementary angle is defined to be the complementary secant (cosecant) of the angle.

Review:



$$\sin\theta := \frac{Op}{H} \qquad \cos\theta := \frac{A}{H}$$

$$\tan\theta := \frac{Op}{A} \qquad \cot\theta := \frac{A}{Op}$$

$$\sec\theta := \frac{H}{A} \qquad \csc\theta := \frac{H}{Op}$$

Facts:

$$\cos\theta = \sin(\pi/2 - \theta)$$

$$\cot\theta = \tan(\pi/2 - \theta)$$

$$\csc\theta = \sec(\pi/2 - \theta)$$

Since θ is the complementary angle to $\pi/2 - \theta$, we also have:

$$\boxed{\begin{aligned} \sin\theta &= \cos(\pi/2 - \theta) \\ \tan\theta &= \cot(\pi/2 - \theta) \\ \sec\theta &= \csc(\pi/2 - \theta) \end{aligned}}$$

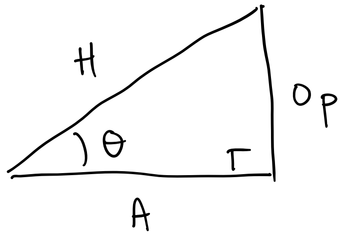
From the definitions, we also have

$$\boxed{\sec\theta = \frac{1}{\cos\theta}, \quad \csc\theta = \frac{1}{\sin\theta}, \quad \cot\theta = \frac{1}{\tan\theta}}$$

Equivalently:

$$\boxed{\cos\theta = \frac{1}{\sec\theta}, \quad \sin\theta = \frac{1}{\csc\theta}, \quad \tan\theta = \frac{1}{\cot\theta}}$$

Now from the right triangle:



$$\boxed{A^2 + Op^2 = H^2}$$

by Pythagorean Theorem.

Dividing by H^2 gives

$$\left(\frac{A}{H}\right)^2 + \left(\frac{Op}{H}\right)^2 = 1.$$ From the

definitions, we have

$$\boxed{\sin^2\theta + \cos^2\theta = 1} \quad (1)$$

Dividing by $\sin^2\theta$ gives us

$$1 + \left(\frac{\cos\theta}{\sin\theta}\right)^2 = \frac{1}{\sin^2\theta}$$

Notice that $\dfrac{\cos\theta}{\sin\theta} = \dfrac{A/H}{Op/H} = \dfrac{A}{Op} = \cot\theta$,

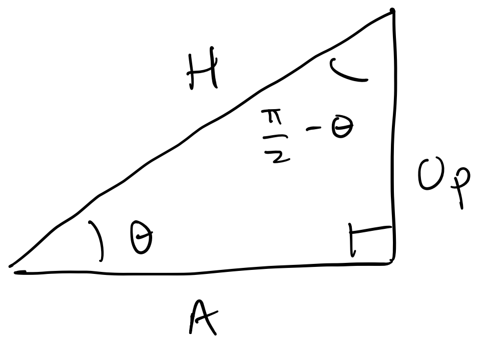
so $$\boxed{1 + \cot^2\theta = \csc^2\theta}$$

Dividing (1) by $\cos^2\theta$ gives us

$$\left(\frac{\sin\theta}{\cos\theta}\right)^2 + 1 = \frac{1}{\cos^2\theta}.$$

Since $\dfrac{\sin\theta}{\cos\theta} = \dfrac{Op/H}{A/H} = \dfrac{Op}{A} = \tan\theta$, we have that $\boxed{\tan^2\theta + 1 = \sec^2\theta}$.

Review:



$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta}$$

$$\sin^2\theta + \cos^2\theta = 1$$

$$1 + \cot^2\theta = \csc^2\theta$$

$$\tan^2\theta + 1 = \sec^2\theta$$

$$\sec\theta = \frac{1}{\cos\theta}$$

$$\csc\theta = \frac{1}{\sin\theta}$$

$$\cot\theta = \frac{1}{\tan\theta}$$

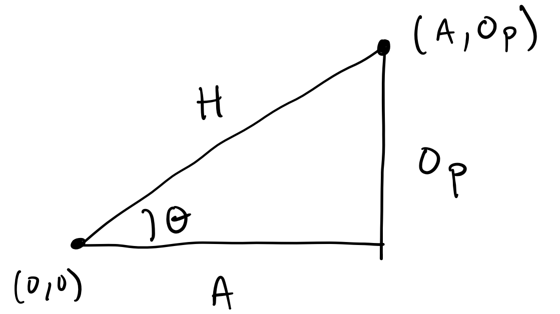
$$\sin\theta = \cos(\pi/2 - \theta)$$

$$\sec\theta = \csc(\pi/2 - \theta)$$

$$\tan\theta = \cot(\pi/2 - \theta)$$

All of our results were for positive, acute angles.

Let's extend our results to hold for <u>any</u> angle.



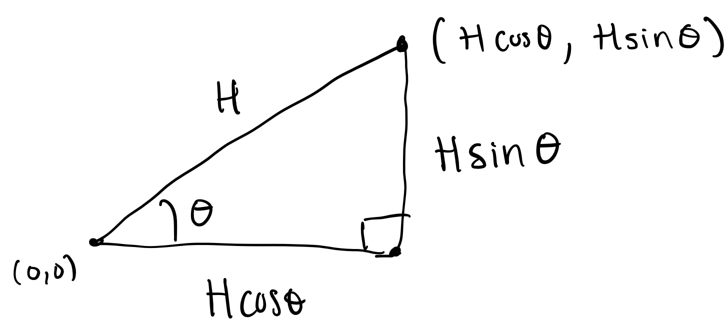
$$\sin\theta = \frac{Op}{H} \longrightarrow$$

$$H\sin\theta = Op$$

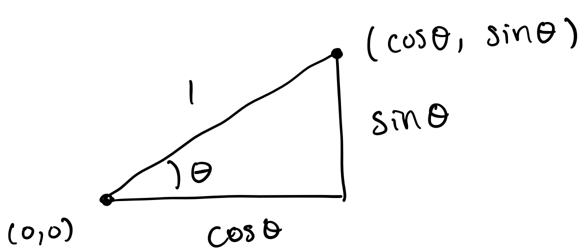
$$\cos\theta = \frac{A}{H} \longrightarrow$$

$$H\cos\theta = A$$

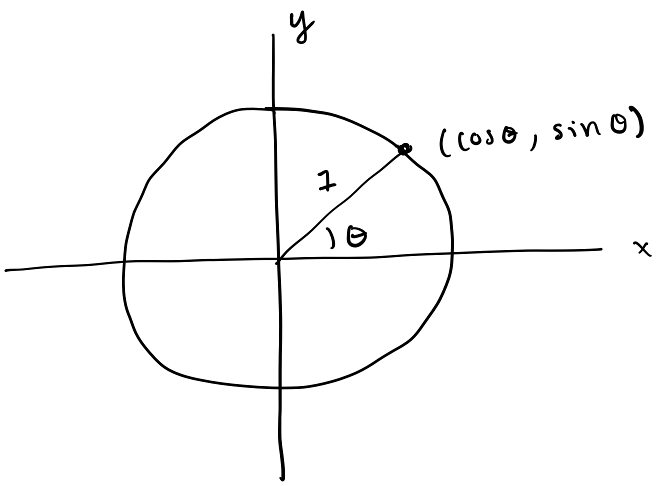
Thus this triangle satisfies:



Let's set $H = 1$; then



The point $(\cos\theta, \sin\theta)$ lies on the circle of radius 1 that is centered at the origin (aka the unit circle).



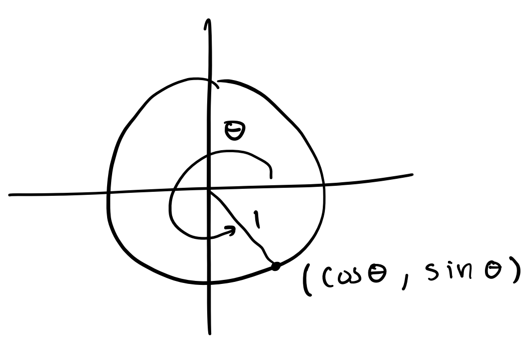
This tells us how we can extend the definition of $\cos\theta$, $\sin\theta$ to angles that are not acute:

For any angle θ

$\cos\theta$ is the x-coordinate, and

$\sin\theta$ is the y-coordinate, of

the point on the unit circle that corresponds to the angle θ, e.g.



We now need to extend tan, sec, csc, cot. To be consistent with the results we found earlier, we define

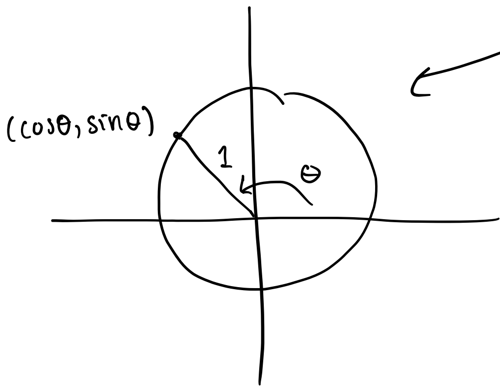
$$\boxed{\sec\theta := \frac{1}{\cos\theta}, \quad \csc\theta := \frac{1}{\sin\theta}, \quad \cot\theta := \frac{\cos\theta}{\sin\theta}, \text{ and } \quad \tan\theta := \frac{\sin\theta}{\cos\theta}}$$

These definitions now allow us to use <u>any</u> angle θ.

FACT: <u>ALL</u> of the results we found for acute angles <u>still</u> hold with our extended definitions.

Review :

For any angle θ :



$\sin\theta$, $\cos\theta$ defined by

$$\tan\theta := \frac{\sin\theta}{\cos\theta}$$

$$\sec\theta := \frac{1}{\cos\theta} , \quad \csc\theta := \frac{1}{\sin\theta}$$

$$\cot\theta := \frac{\cos\theta}{\sin\theta}$$

$$\tan\theta = \frac{1}{\cot\theta} ,$$

$$\cot\theta = 1/\tan\theta$$

$$\sin^2\theta + \cos^2\theta = 1$$

$$1 + \cot^2\theta = \csc^2\theta$$

$$\tan^2\theta + 1 = \sec^2\theta$$

$$\tan(\pi/2 - \theta) = \cot\theta$$

$$\cot(\pi/2 - \theta) = \tan\theta$$

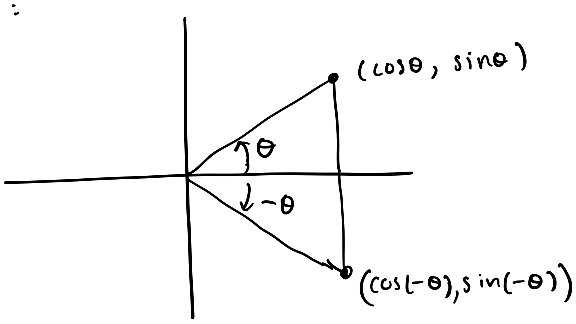
$$\sin(\pi/2 - \theta) = \cos(\theta)$$

$$\cos(\pi/2 - \theta) = \sin\theta$$

$$\sec(\pi/2 - \theta) = \csc(\theta)$$

$$\csc(\pi/2 - \theta) = \sec\theta$$

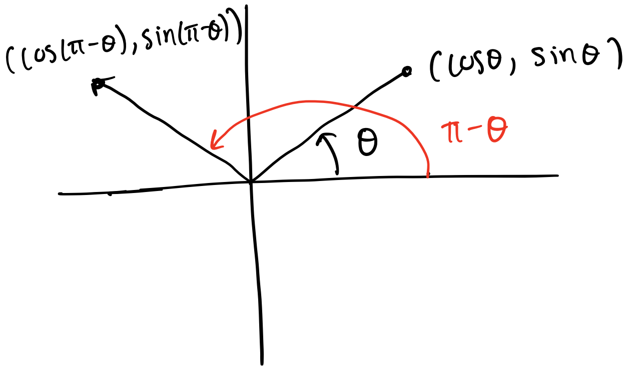
Now that we are working with the unit circle instead of right triangles, we can say even more :



For any angle θ :

$$\cos(-\theta) = \cos\theta$$

$$\sin(-\theta) = -\sin(\theta)$$



For **any** angle θ:

$$\cos(\pi-\theta) = -\cos\theta$$
$$\sin(\pi-\theta) = \sin\theta$$

From our previous results:

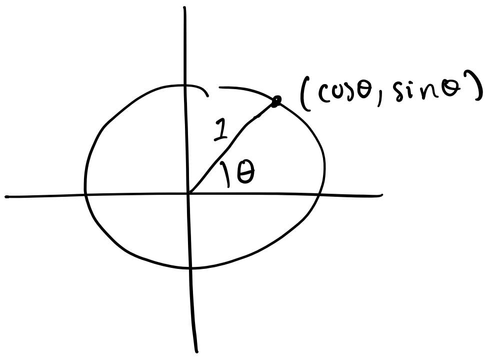
$$\tan(-\theta) = \frac{\sin(-\theta)}{\cos(-\theta)} = \frac{-\sin\theta}{\cos\theta} = -\tan\theta$$

$$\tan(\pi-\theta) = \frac{\sin(\pi-\theta)}{\cos(\pi-\theta)} = \frac{\sin\theta}{-\cos\theta} = -\tan(\theta)$$

Exercise: Show that

$$\cot(-\theta) = -\cot\theta \quad , \quad \cot(\pi-\theta) = -\cot(\theta)$$
$$\sec(-\theta) = \sec\theta \quad , \quad \sec(\pi-\theta) = -\sec(\theta)$$
$$\csc(-\theta) = -\csc\theta \quad , \quad \csc(\pi-\theta) = \csc(\theta)$$

Review: For any angle θ



$\sin\theta$, $\cos\theta$ defined this way

$$\tan\theta := \frac{\sin\theta}{\cos\theta}, \quad \sec\theta := \frac{1}{\cos\theta}$$

$$\cot\theta := \frac{\cos\theta}{\sin\theta}, \quad \csc\theta := \frac{1}{\sin\theta}$$

$$\tan\theta = \frac{1}{\cot\theta}, \quad \cot\theta = \frac{1}{\tan\theta}$$

$$\sin^2\theta + \cos^2\theta = 1$$
$$1 + \cot^2\theta = \csc^2\theta$$
$$\sec^2\theta + 1 = \tan^2\theta$$

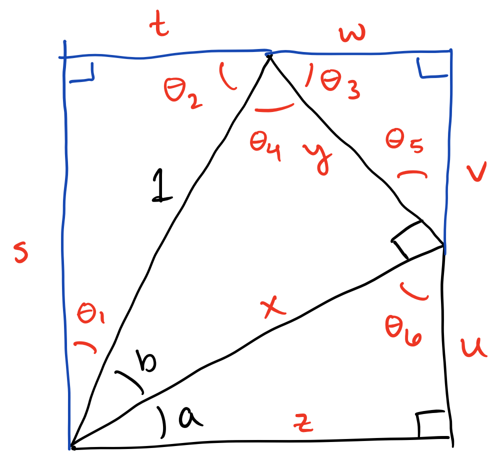
$$\sin(\pi/2 - \theta) = \cos\theta$$
$$\cos(\pi/2 - \theta) = \sin\theta$$
$$\tan(\pi/2 - \theta) = \cot\theta$$
$$\cot(\pi/2 - \theta) = \tan\theta$$
$$\sec(\pi/2 - \theta) = \csc(\theta)$$
$$\csc(\pi/2 - \theta) = \sec(\theta)$$

$$\sin(\pi - \theta) = -\sin\theta$$
$$\cos(\pi - \theta) = \cos\theta$$
$$\tan(\pi - \theta) = -\tan\theta$$
$$\cot(\pi - \theta) = -\cot\theta$$
$$\csc(\pi - \theta) = \csc(\theta)$$
$$\sec(\pi - \theta) = -\sec\theta$$

$$\sin(-\theta) = -\sin\theta$$
$$\cos(-\theta) = \cos\theta$$
$$\tan(-\theta) = -\tan\theta$$
$$\cot(-\theta) = -\cot\theta$$
$$\sec(-\theta) = \sec\theta$$
$$\csc(-\theta) = -\csc\theta$$

Let's revisit triangles: Let a, b be acute angles.

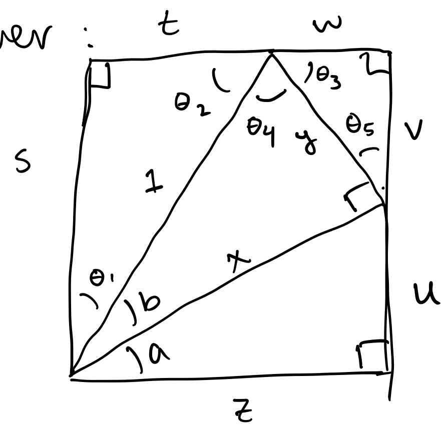
Consider the following:



(HARD)

Exercise: Find the values of all unknowns, labeled in red, in terms of a, b, sine, and cosine.

Answer:



$\theta_1 = \pi/2 - (a+b)$

$\theta_2 = a+b$

$x = \cos b$

$y = \sin b$

$z = \cos b \cos a$

$u = \cos b \sin a$

$\theta_4 = \pi/2 - b$

$\theta_3 = \pi - \theta_2 - \theta_4 = \pi/2 - a$

$w = y \sin a$

$= \sin b \sin a$

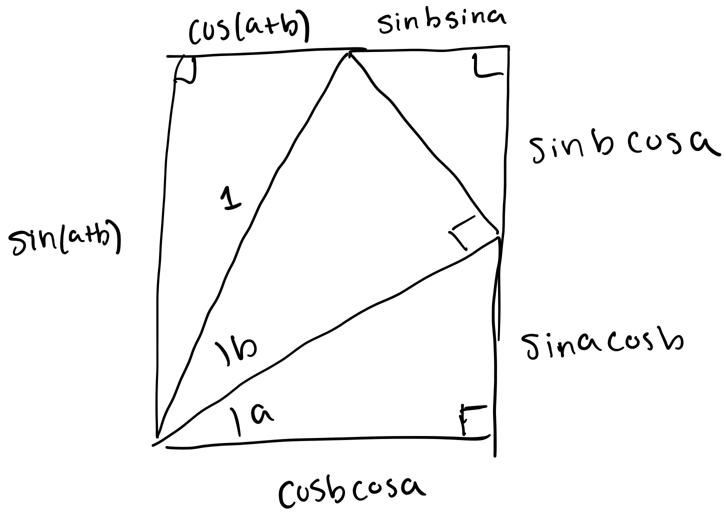
$\theta_5 = a$

$v = y \cos a = \sin b \cos a$

$s = \sin(a+b)$

$t = \cos(a+b)$

In other words:



We then have:

$$\boxed{\begin{aligned} \cos(a+b) &= \cos a \cos b - \sin a \sin b \\ \sin(a+b) &= \sin a \cos b + \sin b \cos a \end{aligned}}$$

Remark: These formulas hold for <u>any</u> angle a and b, no matter if they are acute or not. Just trust me on this.

From these:

$$\begin{aligned} \sin(a-b) &= \sin(a)\cos(-b) + \sin(-b)\cos a \\ &= \sin(a)\cos b - \sin b \cos a \end{aligned}$$

$$\begin{aligned} \cos(a-b) &= \cos a \cos(-b) - \sin a \sin(-b) \\ &= \cos a \cos b + \sin a \sin b \end{aligned}$$

Review:

$$\boxed{\begin{aligned} &(2)\quad \sin(a+b) = \sin a\cos b + \sin b\cos a \\ &(3)\quad \sin(a-b) = \sin a\cos b - \sin b\cos a \\ &(4)\quad \cos(a+b) = \cos a\cos b - \sin a\sin b \\ &(5)\quad \cos(a-b) = \cos a\cos b + \sin a\sin b \end{aligned}}$$

Now adding equations 2 and 3 gives us

$$\boxed{\sin(a+b) + \sin(a-b) = 2\sin a\cos b}$$

subtracting gives us

$$\boxed{\sin(a+b) - \sin(a-b) = 2\sin b\cos a}$$

Adding equations 4 and 5 gives us

$$\boxed{\cos(a+b) + \cos(a-b) = 2\cos a\cos b}$$

Subtracting equation 4 from equation 5 gives us

$$\boxed{\cos(a-b) - \cos(a+b) = 2\sin a \sin b}$$

Review :

$$2\sin a\cos b = \sin(a+b) + \sin(a-b)$$

$$2\cos a\sin b = \sin(a+b) - \sin(a-b)$$

$$2\cos a\cos b = \cos(a+b) + \cos(a-b)$$

$$2\sin a\sin b = \cos(a-b) - \cos(a+b)$$

Let's play around with these some more.

Let $a = \frac{u+v}{2}$ and $b = \frac{u-v}{2}$. Then

$a+b = u$ and $a-b = v$. Putting these into the formulas above gives us

$$2\sin\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right) = \sin u + \sin v$$

$$2\cos\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right) = \sin u - \sin v$$

$$2\cos\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right) = \cos u + \cos v$$

$$2\sin\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right) = \cos v - \cos u$$

Review:

$$\begin{cases} \sin(a+b) = \sin a\cos b + \cos a\sin b \\ \sin(a-b) = \sin a\cos b - \cos a\sin b \\ \cos(a+b) = \cos a\cos b - \sin a\sin b \\ \cos(a-b) = \cos a\cos b + \sin a\sin b \end{cases}$$

$$\begin{cases} 2\sin a\cos b = \sin(a+b) + \sin(a-b) \\ 2\cos a\sin b = \sin(a+b) - \sin(a-b) \\ 2\cos a\cos b = \cos(a+b) + \cos(a-b) \\ 2\sin a\sin b = \cos(a-b) - \cos(a+b) \end{cases}$$

$$\begin{cases} \sin(u) + \sin(v) = 2\sin\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right) \\ \sin(u) - \sin(v) = 2\cos\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right) \\ \cos(u) + \cos v = 2\cos\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right) \\ \cos(v) - \cos(u) = 2\sin\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right) \end{cases}$$

Now let's re-visit

$$\sin(a+b) = \sin a \cos b + \cos a \sin b$$

$$\cos(a+b) = \cos a \cos b - \sin a \sin b$$

Let $a = b = u$. Then

$$\boxed{\begin{aligned} \sin(2u) &= 2\sin u \cos u \\ \cos(2u) &= \cos^2 u - \sin^2 u \end{aligned}} \quad (6)$$

Recall $\sin^2 u + \cos^2 u = 1$. Therefore

$$\boxed{\begin{aligned} \sin^2 u &= 1 - \cos^2 u \quad (7) \\ \cos^2 u &= 1 - \sin^2 u \quad (8) \end{aligned}}$$

Substituting equation 7 into equation 6

gives us

$$\boxed{\cos 2u = 2\cos^2 u - 1}$$

substituting equation 8 into equatoon 6

gives us

$$\boxed{\cos 2u = 1 - 2\sin^2 u}$$

Review:

$$\boxed{\begin{aligned} \sin(2u) &= 2\sin u \cos u \\ \cos(2u) &= \cos^2 u - \sin^2 u \\ &= 2\cos^2 u - 1 \\ &= 1 - 2\sin^2 u \end{aligned}}$$

From these:

$$\tan(2u) = \frac{\sin 2u}{\cos 2u} = \frac{2\sin u \cos u}{\cos^2 u - \sin^2 u}$$

$$= \frac{2\sin u \cos u}{\cos^2 u \left(1 - \frac{\sin^2(u)}{\cos^2(u)}\right)} = \frac{2\,\frac{\sin u}{\cos u}}{1 - \frac{\sin^2 u}{\cos^2 u}} = \frac{2\tan u}{1 - \tan^2 u}$$

Review:

$$\boxed{\tan 2u = \frac{2\tan u}{1-\tan^2 u}}$$

Now let's play around with

$\cos(2u) = 2\cos^2 u - 1.$ Then

$$\boxed{\frac{\cos(2u)+1}{2} = \cos^2 u}$$

If we start with

$\cos(2u) = 1-2\sin^2 u,$ then

$$\boxed{\sin^2 u = \frac{1-\cos(2u)}{2}}$$

Then $\tan^2 u = \dfrac{\sin^2 u}{\cos^2 u} = \dfrac{\frac{1-\cos(2u)}{2}}{\frac{1+\cos(2u)}{2}}$

$$= \frac{1-\cos(2u)}{1+\cos(2u)}$$

i.e. $\boxed{\tan^2 u = \dfrac{1-\cos(2u)}{1+\cos(2u)}}$

From this:

$$\tan^2 u = \frac{1-\cos(2u)}{1+\cos(2u)} \cdot \frac{1-\cos(2u)}{1-\cos(2u)} = \frac{(1-\cos(2u))^2}{1-\cos^2(2u)}$$

$$= \frac{(1-\cos(2u))^2}{\sin^2(2u)} = \left(\frac{1-\cos(2u)}{\sin(2u)}\right)^2$$

so $\boxed{\tan^2 u = \left(\dfrac{1-\cos(2u)}{\sin(2u)}\right)^2}$

Also: $\tan^2 u = \dfrac{1-\cos(2u)}{1+\cos(2u)} \cdot \dfrac{1+\cos(2u)}{1+\cos(2u)} = \dfrac{1-\cos^2(2u)}{(1+\cos(2u))^2}$

$$= \frac{\sin^2(2u)}{(1+\cos(2u))^2} = \left(\frac{\sin(2u)}{1+\cos(2u)}\right)^2$$

Review:

$$\sin(2u) = 2\sin u \cos u$$

$$\begin{aligned}\cos(2u) &= \cos^2 u - \sin^2 u \\ &= 2\cos^2 u - 1 \\ &= 1 - 2\sin^2 u\end{aligned}$$

$$\tan 2u = \frac{1-\cos 2u}{1+\cos 2u}$$

$$\cos^2 u \overset{(9)}{=} \frac{1+\cos(2u)}{2}, \quad \sin^2 u \overset{(10)}{=} \frac{1-\cos(2u)}{2},$$

$$\tan^2 u \overset{(13)}{=} \frac{1-\cos(2u)}{1+\cos(2u)} \overset{(11)}{=} \left(\frac{1-\cos(2u)}{\sin(2u)}\right)^2$$

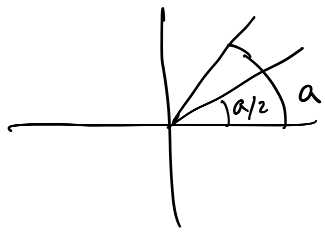
$$\overset{(12)}{=} \left(\frac{\sin(2u)}{1+\cos(2u)}\right)^2$$

In equations 9, 10, 11, 12, if we let $a = 2u$, so that $u = a/2$, then

$$\cos^2(a/2) \overset{(13)}{=} \frac{1+\cos a}{2}, \quad \sin^2(a/2) \overset{(14)}{=} \frac{1-\cos a}{2},$$

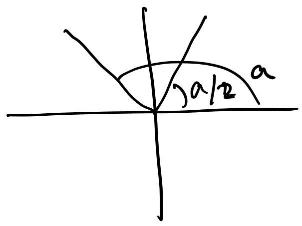
$$\tan^2(a/2) \overset{(15)}{=} \left(\frac{1-\cos a}{\sin a}\right)^2 \overset{(16)}{=} \left(\frac{\sin a}{1+\cos a}\right)^2$$

Fact:



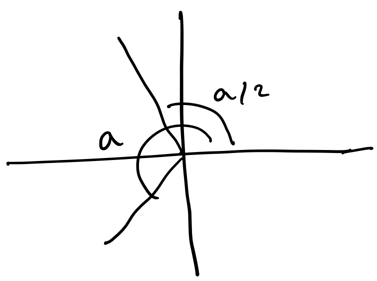
If $0 < a < \pi/2$, then $0 < a/2 < \pi/4$,

so $\sin a > 0$ and $\tan a/2 > 0$.



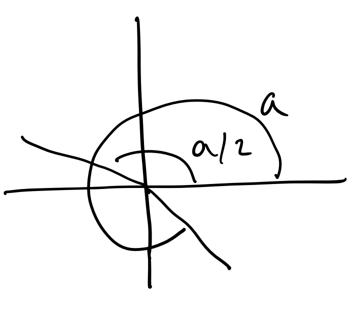
If $\pi/2 < a < \pi$, then $\pi/4 < a/2 < \pi/2$,

so $\sin a > 0$ and $\tan a/2 > 0$.



If $\pi < a < 3\pi/2$, then $\frac{\pi}{2} < a/2 < \frac{3\pi}{4}$,

so $\sin a < 0$ and $\tan a/2 < 0$



If $\frac{3\pi}{2} < a < 2\pi$,

then $\frac{3\pi}{4} < \frac{a}{2} < \pi$,

so $\sin a < 0$ and $\tan a/2 < 0$

In all cases: $\tan \frac{a}{2}$ has the same sign as $\sin a$ for any angle a

why do we care: Let's take square roots of both sides of equations 13, 14, 15, 16:

$$|\cos(a/2)| = \sqrt{\frac{1+\cos a}{2}} \longleftrightarrow \cos(a/2) = \pm\sqrt{\frac{1+\cos a}{2}}$$

$$|\sin(a/2)| = \sqrt{\frac{1-\cos a}{2}} \longleftrightarrow \sin(a/2) = \pm\sqrt{\frac{1-\sin a}{2}}$$

$$|\tan(a/2)| = \left|\frac{1-\cos a}{\sin a}\right| \longleftrightarrow \tan(a/2) = \frac{1-\cos a}{\sin a}$$

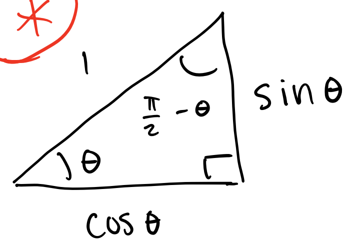
$$|\tan(a/2)| = \left|\frac{\sin a}{1+\cos a}\right| \longleftrightarrow \tan(a/2) = \frac{\sin a}{1+\cos a}$$

The last two results are only true because $\tan(a/2)$ has the same sign as $\sin a$.

We have now shown all of the identities you need to know. Here is a flow chart / cheat sheet on how to study them.

If I mark it in red: you <u>must</u> memorize it. No way around it. Otherwise, it is possible to figure it out without memorization by using <u>only</u> previous things.

1. ✱



$\tan\theta := \dfrac{\sin\theta}{\cos\theta}$ $\qquad \sec\theta := \dfrac{1}{\cos\theta}$

$\cot\theta := \dfrac{\cos\theta}{\sin\theta}$ $\qquad \csc\theta := \dfrac{1}{\sin\theta}$

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2. $\sin(\pi/2 - \theta) = \cos(\theta)$ $\qquad \tan\theta = \dfrac{1}{\cot\theta}$

$\sec(\pi/2 - \theta) = \csc(\theta)$ $\qquad \sin^2\theta + \cos^2\theta = 1$

$\tan(\pi/2 - \theta) = \cot(\theta)$

---

3. $\tan^2\theta + 1 = \sec^2\theta$, $\qquad 1 + \cot^2\theta = \csc^2\theta$

$\sin^2\theta = 1 - \cos^2\theta$, $\qquad \cos^2\theta = 1 - \sin^2\theta$

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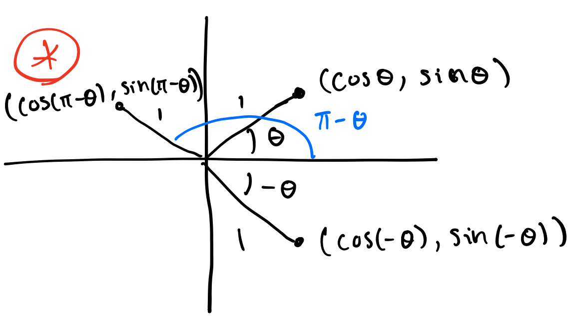
4. ✱



$\cos(-\theta) = \cos\theta$

$\sin(-\theta) = -\sin\theta$

$\cos(\pi - \theta) = -\cos\theta$

$\sin(\pi - \theta) = \sin\theta$

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5. $\tan(\pi-\theta) = -\tan\theta$ $\tan(-\theta) = -\tan\theta$

$\cot(\pi-\theta) = -\cot\theta$ $\cot(-\theta) = -\cot\theta$

$\sec(\pi-\theta) = -\sec\theta$ $\sec(-\theta) = \sec\theta$

$\csc(\pi-\theta) = \csc\theta$ $\csc(-\theta) = -\csc\theta$

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6. (*) $\sin(a+b) = \sin a\cos b + \cos a\sin b$

$\cos(a+b) = \cos a\cos b - \sin a\sin b$

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7. $\sin(a-b) = \sin a\cos b - \cos a\sin b$

$\cos(a-b) = \cos a\cos b + \sin a\sin b$

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8. $\sin(a+b) + \sin(a-b) = 2\sin a\cos b$

$\sin(a+b) - \sin(a-b) = 2\cos a\sin b$

$\cos(a+b) + \cos(a-b) = 2\cos a\cos b$

$\cos(a-b) - \cos(a+b) = 2\sin a\sin b$

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9. $\sin u + \sin v = 2\sin\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right)$

$\sin(u) - \sin v = 2\cos\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right)$

$\cos(u) + \cos v = 2\cos\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right)$

$\cos(u) - \cos(v) = 2\sin\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right)$

10. $\sin(2u) = 2\sin u \cos u$

$\cos(2u) = \cos^2 u - \sin^2 u$

$\cos(2u) = 2\cos^2 u - 1$

$\cos(2u) = 1 - 2\sin^2 u$

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11. $\tan(2u) = \dfrac{2\tan u}{1 - \tan^2 u}$

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12. $\sin^2 u = \dfrac{1 - \cos(2u)}{2}$, $\cos^2(u) = \dfrac{1 + \cos(2u)}{2}$,

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13. $\tan^2 u = \dfrac{1 - \cos(2u)}{1 + \cos(2u)}$

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14. $\sin(u/2) = \pm\sqrt{\dfrac{1 - \cos(2u)}{2}}$,

$\cos(u/2) = \pm\sqrt{\dfrac{1 + \cos(2u)}{2}}$,

$\tan(u/2) = \dfrac{1 - \cos u}{\sin u}$

$\tan(u/2) = \dfrac{\sin u}{1 + \cos u}$

Good luck! You got this!