2D FEM Error Calculation

(Read 2D FEM Notes first!)

L^2 Error

$$\begin{cases} -\Delta u + qu = f & x \in \Omega \subset \mathbb{R}^2 \\ \dfrac{\partial u}{\partial \nu} = g & x \in \partial\Omega \end{cases}$$

Weak form: $\displaystyle\int_\Omega \nabla u \nabla v + quv \, dx = \int_\Omega fv\, dx + \int_{\partial\Omega} gv\, ds$

Approximate solution

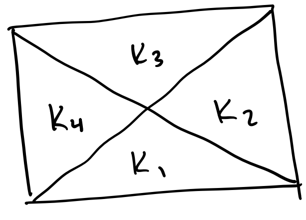
$$u_h = \sum_i u_i \phi_i \, , \quad \{\phi_i\} \text{ basis functions}$$

for 2D Piecewise linear Lagrange FEM space

L^2 error: $\displaystyle \|u - u_h\|_{L^2}^2 = \int_\Omega (u - u_h)^2 \, dx$

Let $\{K_\ell\}$ be the collection of mesh elements

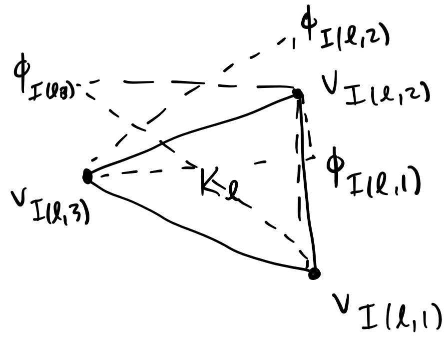
eg



Let $I(\ell, j)$ be the local-to-global vertex enumeration for the cells

Then $\|u-u_h\|_{L^2}^2 = \sum_{\ell} \int_{K_\ell} (u-u_h)^2 \, dx$

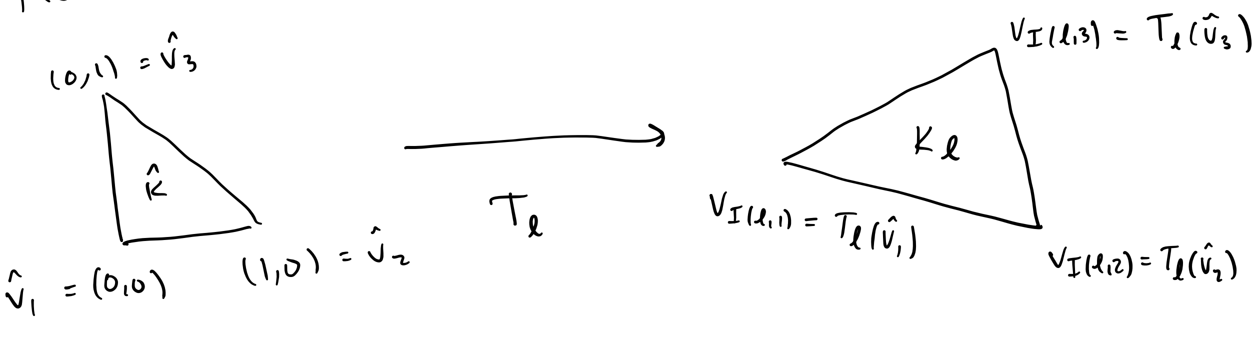
Recall: On a particular element K_ℓ



$$u_h\big|_{K_\ell} = u_{I(\ell,1)}\,\phi_{I(\ell,1)} + u_{I(\ell,2)}\,\phi_{I(\ell,2)} + u_{I(\ell,3)}\,\phi_{I(\ell,3)}$$

Thus $\|u-u_h\|_{L^2}^2 = \sum_{\ell} \int_{K_\ell} \left(u - \sum_{i=1}^{3} u_{I(\ell,i)}\,\phi_{I(\ell,i)}\right)^2 dx$

Recall the reference map for cells:



with properties: 1. $T_\ell(\hat{x}) = v_{I(\ell,1)} + \underbrace{\left[\, v_{I(\ell,2)} - v_{I(\ell,1)} \;\middle|\; v_{I(\ell,3)} - v_{I(\ell,1)} \,\right]}_{B_\ell} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix}$

2. $DT_\ell = B_\ell$

3. $\phi_{I(\ell,i)} \circ T_\ell = \hat{\phi}_i$

4. $\hat{\phi}_1(\hat{x}) = 1 - \hat{x}_1 - \hat{x}_2$

$\hat{\phi}_2(\hat{x}) = \hat{x}_1$

$\hat{\phi}_3(\hat{x}) = \hat{x}_2$

Thus $$\|u-u_n\|_{L^2}^2 = \sum_{\ell} \int_{\hat{K}} \Big(u \circ T_\ell - \sum_i u_{I(\ell,i)} \hat{\phi}_i\Big)^2 \underbrace{|\det B_\ell|}_{|\det DT_\ell|}\, d\hat{x}$$

$x = T_\ell(\hat{x})$ change of variable

Recall quadrature rules on cells:

$$\int_{\hat{K}} f(\hat{x})\, d\hat{x} \approx \text{Area}(\hat{K}) \sum_j \hat{w}_j f(\hat{x}_j) = \frac{1}{2} \sum_j \hat{w}_j f(\hat{x}_j)$$

Thus

$$\boxed{\|u-u_n\|_{L^2}^2 \approx \sum_{\ell=1}^{\#\text{cells}} \frac{|\det B_\ell|}{2} \sum_{j=1}^{\#\text{quadrature pts}} \hat{w}_j \Big(u(T_\ell(\hat{x}_j)) - \sum_{i=1}^{3} u_{I(\ell,i)} \hat{\phi}_i(\hat{x}_j)\Big)^2}$$

Take square roots to get L^2 error.

H^1 Seminorm Error (aka L^2 norm of gradient)

$$|u-u_h|_{H^1}^2 = \|\nabla u - \nabla u_h\|_{L^2}^2$$

$$= \int_\Omega |\underbrace{\nabla u - \nabla u_h}_{\text{Euclidean norm (squared) of \underline{vector} } \nabla u - \nabla u_h}|^2 \, dx$$

$$= \sum_\ell \int_{K_\ell} \Big|\nabla u - \nabla\Big(\sum_i u_{I(\ell,i)} \phi_{I(\ell,i)}\Big)\Big|^2 dx$$

$$= \sum_\ell \int_{K_\ell} \Big|\nabla u - \sum_i u_{I(\ell,i)} \nabla \phi_{I(\ell,i)}\Big|^2 dx$$

Recall change of variables for gradients:

$$\nabla \underbrace{(\phi_{I(\ell,i)} \circ T_\ell)}_{\hat{\phi}_i} \underset{\text{Chain rule}}{=} (\nabla \phi_{I(\ell,i)}) \circ T_\ell \; \underbrace{DT_\ell}_{B_\ell} \rightarrow$$

$$\underbrace{(\nabla_{\hat{x}} \hat{\phi}_i)}_{\text{row vector}} \underbrace{B_\ell^{-1}}_{\text{matrix multiplication}} = (\nabla \phi_{I(\ell,i)}) \circ T_\ell$$

Therefore:

$$|u-u_n|^2_{H^1} = \sum_{\ell} \int_{\hat{K}} \left| \nabla_x u \circ T_\ell - \sum_i u_{I(\ell,i)} (\nabla_{\hat{x}} \hat{\phi}_i) B_\ell^{-1} \right|^2 |\det B_\ell| \, d\hat{x}$$

$\uparrow$

$x = T_\ell(\hat{x})$ change of variable

$\longrightarrow$

$$\boxed{|u-u_n|^2_{H^1} \approx \sum_{\ell=1}^{\#\text{cells}} \frac{|\det B_\ell|}{2} \sum_{j=1}^{\#\text{quad pts}} \hat{w}_j \left| \nabla_x u(T_\ell(\hat{x}_j)) - \left(\sum_{i=1}^{3} u_{I(\ell,i)} \nabla_{\hat{x}} \hat{\phi}_i(\hat{x}_j) \right) B_\ell^{-1} \right|^2}$$

Either take square roots to get the L^2 error of the gradient, or if you want the full H^1 error:

$$\|u-u_n\|^2_{H^1} = \|u-u_n\|^2_{L^2} + |u-u_n|^2_{H^1}$$

Then take square roots.