HW 4 Hints

1. Your answer should be of the form:

Find $u \in V$ such that

$$\left| \begin{array}{l} -(ku')' + q u = f \quad \text{on} \quad (0,1), \\ \text{Boundary condition at } x=0, \\ \text{Boundary condition at } x=1. \end{array} \right.$$

Find V, q, and the boundary conditions.

For q, consider a piecewise-defined function.

2. Continuity of a: Show there is a C_1 st for all u, φ,

$$|a(u,\varphi)| \leq C_1 \sqrt{\int_\Omega (|\nabla u|^2 + |u|^2)} \sqrt{\int_\Omega (|\nabla \varphi|^2 + |\varphi|^2)}$$

Ellipticity: Show $\exists\, C_2 > 0$ st for all u,

$$a(u,u) \geq C_2 \int_\Omega (|\nabla u|^2 + |u|^2)$$

Continuity of l: show $\exists\, C_3$ st for all φ,

$$|l(\varphi)| \leq C_3 \sqrt{\int_\Omega |\varphi|^2}$$

The following facts may be useful:

1. For any F

$$\left| \int_\Omega F \right| \le \int_\Omega |F|$$

2. For any F_1, F_2

$$|F_1 + F_2| \le |F_1| + |F_2|$$

3. For any u, φ

$$|\nabla u \cdot \nabla \varphi| \le |\nabla u| \, |\nabla \varphi|$$

4. For any c, F

$$|cF| = |c| \, |F|$$

5. There is a constant $C_\Omega > 0$ such that, for all $u \in H_0^1(\Omega)$,

$$\int_\Omega |\nabla u|^2 \ge C_\Omega \int_\Omega |u|^2$$

6. For any c, F

$$\int_\Omega |c| \, |F| \le \sup_{x \in \Omega} |c(x)| \int_\Omega |F|$$

⑦ For any x_1, x_2, y_1, y_2

$$|x_1 y_1 + x_2 y_2| \leq \sqrt{x_1^2 + x_2^2}\,\sqrt{y_1^2 + y_2^2}$$

⑧ For any F, G,

$$\int_\Omega |F||G| \leq \sqrt{\int_\Omega |F|^2}\,\sqrt{\int_\Omega |G|^2}$$

3. A real inner product on a real vector space V is a map $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ such that

(1) $\langle c u_1 + u_2, v \rangle = c\langle u_1, v\rangle + \langle u_2, v\rangle$ for all $c \in \mathbb{R}$, $u_1, u_2, v \in V$

(2) $\langle u, v \rangle = \langle v, u \rangle$ for all $u, v \in V$

(3) $\langle u, u \rangle \geq 0$ for all $u \in V$

(4) If $u \neq 0$, then $\langle u, u \rangle > 0$.

Real inner products also have the following properties:

(1) $\langle u, c v_1 + v_2 \rangle = c\langle u, v_1\rangle + \langle u, v_2 \rangle$ for all $c \in \mathbb{R}$, $u, v_1, v_2 \in V$

(2) $\langle 0, v \rangle = 0 = \langle u, 0 \rangle$ for all $u, v \in V$

For (a), compute $\langle u+v, u+v\rangle + \langle u-v, u-v\rangle$

For (b), consider the quadratic polynomial

$$p(t) := \langle u+tv, u+tv\rangle \overset{\text{why?}}{=} \|u\|^2 + 2t\langle u,v\rangle + t^2\|v\|^2 .$$

We have $p(t) \geq 0$ for all t (why?)

Use calc I facts to find where the mininum of p is at. Suppose it is at t_0. Compute $p(t_0)$, which satisfies $p(t_0) \geq 0$, and then conclude.

For (c) and (d), recall that the H_0^1 inner product is

$$\langle u, \varphi\rangle_{H_0^1(\Omega)} = \int_\Omega (\nabla u \cdot \nabla \varphi + u\varphi).$$

You should use the results of question 2.