1a. We apply the following strategy to the problem:

1. Suppose we find a sequence $u_n \in C_0^1(-1,1)$. Then since $u(-1) = u(1) = 0$ and $u_n(-1) = u_n(1) = 0$, we have $u - u_n \in H_0^1(-1,1)$. Poincaré then implies that there is a constant $C > 0$ such that

$$\|u-u_n\|_{H^1}^2 = \int_{-1}^{1} (u-u_n)^2 + \int_{-1}^{1} (u'-u_n')^2$$

$$\leq (C+1) \int_{-1}^{1} (u'-u_n')^2 .$$

Thus $\|u-u_n\|_{H^1} \leq \sqrt{C+1}\, \|u'-u_n'\|_{L^2}$, so we only need to find a sequence $u_n \in C_0^1(-1,1)$ for which $\|u'-u_n'\|_{L^2} \to 0$ as $n \to \infty$.

2. Now suppose we find a sequence $v_n \in C_0(-1,1)$ (only continuous, not necessarily differentiable) such that $\|u' - v_n\|_{L^2} \to 0$ as $n \to \infty$. Then if we let $u_n(x) = \int_{-1}^{x} v_n(s)\, ds$, we have $u_n \in C_0^1(-1,1)$,

$u_n' = v_n$, and, by (1.), $\|u - u_n\|_{H^1} \to 0$ as $n \to \infty$. Thus we only need to find a sequence $v_n \in C_0(-1,1)$ of continuous functions that approximates u' in the sense that $\|u' - v_n\|_{L^2} \to 0$ as $n \to \infty$.

3. We can reduce the problem further using the following theorem:

<u>The Dominated Convergence Theorem for $L^2(-1,1)$</u>:

Let (f_n) be a sequence of functions in $L^2(-1,1)$, and let $f \in L^2(-1,1)$. If

(1) $f_n(x) \to f(x)$ for almost every $x \in (-1,1)$

(2) There is a function $g \in L^2(-1,1)$ for which $|f_n(x)| \leq g(x)$ for almost every $x \in (-1,1)$

Then $\|f_n - f\|_{L^2} \to 0$ as $n \to \infty$.

If we apply this theorem to $f = u'$, $g = |u'|$, then we only need to find a sequence $(v_n) \subset C_0(-1,1)$ for which

1. $|v_n(x)| \leq |u'(x)|$ for almost every $x \in (-1,1)$

2. $v_n(x) \to u'(x)$ for almost every $x \in (-1,1)$.

If we find such a (v_n), then the theorem above tells us $\|v_n - u'\|_{L^2} \to 0$ as $n \to \infty$, and our remarks in (1.), (2.) tell us that the sequence (u_n) with $u_n(x) = \int_{-1}^{x} v_n(t)\,dt$ belongs to $C_0^1(-1,1)$ and $\|u_n - u\|_{H^1} \to 0$ as $n \to \infty$.

To summarize: All you have to do is find a sequence (v_n) w/ $v_n \in C_0(-1,1)$ and

1. $|v_n(x)| \leq |u'(x)|$ for almost every $x \in (-1,1)$

2. $v_n(x) \to u'(x)$ for almost every $x \in (-1,1)$

1b. Compute $\int_{-1}^{0} |u'(x)|dx + \int_0^1 |u'(x)|\, dx$ and

use integration-by-parts on

$\int_{-1}^{0} u\varphi' + \int_0^1 u\varphi'$.

2a. $\|u\|_{H^2}$ should involve u, u', u''.

2b. No hints here. We've done these kinds of problems before.

2c. For boundedness, you need the following inequality:

$$\boxed{\text{For any } u \in H^2(0,1), \text{ any } x_0 \in [0,1], \quad |u'(x_0)| \le \sqrt{2}\, \|u\|_{H^2}.}$$

proof:

$$u'(x_0) - u'(x) = \int_x^{x_0} u''(t)\,dt \quad \xrightarrow{\text{(triangle inequality)}}$$

$$|u'(x_0)| \le |u'(x)| + \int_0^1 |u''(t)|\,dt \quad \xrightarrow{\left(\int_0^1 \cdot\, dx \text{ both sides}\right)}$$

$$|u'(x_0)| \le \int_0^1 |u'(x)|\cdot 1\,dx + \int_0^1 |u''(t)|\cdot 1\,dt$$

$$\overset{\text{(Cauchy-Schwarz)}}{\le} 1\cdot\sqrt{\int_0^1 u'(x)^2\,dx} + 1\cdot\sqrt{\int_0^1 u''(t)^2\,dt}$$

(Cauchy-Schwarz $\mathbb{R}^2$) $\leq \sqrt{2}\sqrt{\int_0^1 (u')^2+(u'')^2} \leq \sqrt{2}\,\|u\|_{H^2}$

def'n of $\|\cdot\|_{H^2}$ $\square$

For ellipticity, you need the following inequality:

$$\boxed{\begin{array}{l}\text{For any } u\in H^2(0,1),\ \text{any } x_0\in[0,1] \\ \displaystyle\int_0^1 (u')^2\,dx \leq 2u'(x_0)^2 + 2\int_0^1 (u'')^2\,dx\end{array}}$$

Proof.

$$u'(x)-u'(x_0) = \int_{x_0}^{x} u''(t)\,dt \longrightarrow$$

$$u'(x) = u'(x_0) + \int_{x_0}^{x} u''(t)\,dt \longrightarrow$$

$$|u'(x)| \overset{\text{triangle}}{\leq} |u'(x_0)| + \int_0^1 |u''(t)|\,dt$$

$$\overset{\text{Cauchy-Schwarz}}{\leq} |u'(x_0)| + \sqrt{\int_0^1 (u'')^2\,dt} \longrightarrow$$

$(a+b)^2 \leq 2a^2+2b^2$

$$u'(x)^2 \leq 2u'(x_0)^2 + 2\int_0^1 u''(t)^2\,dt \xrightarrow{(\int_0^1 dx \text{ both sides})}$$

$$\int_0^1 u'(x)^2 dx \leq 2u'(x_0)^2 + 2\int_0^1 u''(t)^2 dt. \qquad \square$$

2d. Use the hint for boundedness from 2c.

2e. Lax-Milgram for existence. Prove uniqueness directly for full credit: If $u_1, u_2 \in V$ are two solutions to the weak problem, use ellipticity of a to argue that $u_1 = u_2$. As a further hint,

$$\begin{aligned} a(u_1, v) &= \ell(v) \quad \text{for all } v \in V \\ a(u_2, v) &= \ell(v) \end{aligned} \xrightarrow{\text{(bilinearity } a\text{)}}$$

$$a(u_1 - u_2, v) = a(u_1, v) - a(u_2, v) = \ell(v) - \ell(v) = 0$$

for <u>all</u> $v \in V$.