MATH 610, Lecture 600 Nov 3, 2022
Fall 2022

Midterm II Info

Question I.

We have covered the following topics:

- A simple Finite Element method, examples in 1D and 2D
- Finite element spaces, unisolvence
- Implementational aspects (quadrature, assembly)
- a priori error analysis: Quasi optimality, interpolation estimates, the Bramble-Hilbert Lemma

Question II.

Answer the following questions:

a – Recall the definition of Sobolev spaces. What is a generalized derivative? What are the norms of the spaces $H^1(\Omega)$, $H^2(\Omega)$, $H^1_0(\Omega)$ and $L^2(\Omega)$ for $\Omega \subset \mathbb{R}^2$?

b – Also write out the semi norms $|\,.\,|_{H^1(\Omega)}$, $|\cdot|_{H^2(\Omega)}$.

c – What is the Lagrange nodal basis of P_1?

d – Explain in detail how the elliptic model problem, $-\Delta u = f$, $u\Big|_{\partial\Omega} = 0$, on the unit square $\Omega = [0,1]^2$ can be approximated with linear (Lagrange) finite elements. How is the stiffness matrix A and the load vector F constructed?

e – Find unisolvent degrees of freedom for $P_2(T)$, $P_3(T)$ and $P_4(T)$.

f – Give an example of a non-conforming, an H^1-conforming and an H^2-conforming finite element.

g – What is quasi-optimality (Céa's lemma); correspondingly when can you expect an *optimality* result to be true?

h – What is Galerkin orthogonality?

i – What is the statement of the Bramble-Hilbert lemma? Recall the interpolation estimate that we derived with the Bramble-Hilbert lemma on the unit simplex.

j – What is form (structural), shape, and size regularity?

Question III.

~~Show that the Argyrus element defined on a triangle T,~~

$$V(T) = P_5(T), \quad p(a_i),\ \nabla p(a_i),\ \nabla^2 p(a_i),\ \partial_n p(m_i),$$

~~is unisolvent. Here, a_i denotes vertices, and m_i midpoints on edges. Show that the Argyrus element is H^2-conforming.~~

Question IV.

Let $\Omega \subset \mathbb{R}^2$ be a bounded, convex polygonal domain and let $\mathcal{T}_h$ be a triangulation of Ω. Let V_h be the linear finite element space subject to $\mathcal{T}_h$. We consider the elliptic problem

$$-\Delta u + \gamma(x)\, u = f(x) \quad \text{in } \Omega, \qquad u(x) = 0 \quad \text{for } x \in \partial\Omega.$$

(a) Formulate the corresponding Galerkin problems (G) with solution u and (G_h) with solution u_h. What is the correct function space V for the above elliptic problem?

(b) State assumptions on the data ($\mathcal{T}_h$, γ, f) under which (i) the problems (G), (G_h) are well-posed, (ii) the solution $u \in H^2$, and (iii) the following a priori error estimate holds true:

$$\left\|u - u_h\right\|_{L^2(\Omega)} + h\left|u - u_h\right|_{H^1(\Omega)} \leq C\, h^2 \left|u\right|_{H^2(\Omega)}.$$

(c) Prove the a priori estimate given in (b)! More precisely, show that

$$\left|u - u_h\right|_{H^1(\Omega)} \leq C\, h \left|u\right|_{H^2(\Omega)},$$

(d) and with the help of the Aubin-Nitsche trick show that

$$\left\|u - u_h\right\|_{L^2(\Omega)} \leq C\, h^2 \left|u\right|_{H^2(\Omega)}.$$

Question V.

Have a look at the following homework questions: Homework IV, Q2; Homework VI, Q1, Q2, Q3; Homework VII, Q1, Q2.

II

a. A function $u : \Omega \to \mathbb{R}$ has a generalized ith partial derivative v if

$$\int_\Omega u \, \partial_i \phi = - \int_\Omega v \phi$$

for all $\phi : \Omega \to \mathbb{R}$ so 1. $\partial_i \phi : \Omega \to \mathbb{R}$ exists

2. $\phi = 0$ on $\partial\Omega$

$$\|u\|_{H^1(\Omega)} = \left(\int_\Omega |\nabla u|^2 + u^2 \, dx \right)^{1/2}$$

$$\|u\|_{H^2(\Omega)} = \left(\int_\Omega \sum_{i \le j} (\partial_{x_i} \partial_{x_j} u)^2 + |\nabla u|^2 + u^2 \, dx \right)^{1/2}$$

$$\|u\|_{H_0^1(\Omega)} = \left(\int_\Omega |\nabla u|^2 \, dx \right)^{1/2}$$

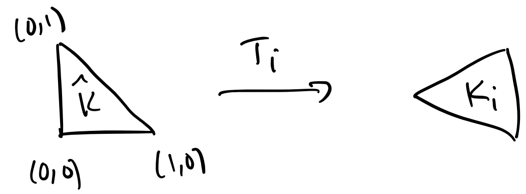
$$\|u\|_{L^2(\Omega)} = \left(\int_\Omega u^2 \, dx \right)^{1/2}$$

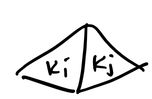
b. $$|u|_{H^1(\Omega)} = \left(\int_\Omega |\nabla u|^2\right)^{1/2}$$

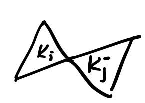
$$|u|_{H^2(\Omega)} = \left(\int_\Omega \sum_{i\le j} (\partial_i \partial_j u)^2 dx\right)^{1/2}$$

c. For simplicity, suppose $\Omega \subset \mathbb{R}^2$ is a polygonal domain. Let $\hat{K}$ be the reference triangle. Suppose that we can construct a triangulation $T_h = \{K_i\}_{i=1,\dots,N}$ of Ω

where 1. Each K_i is a triangle given by an affine reference transformation

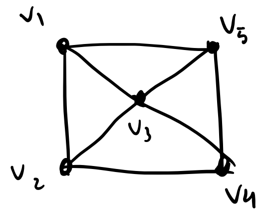


2. $\Omega \subset \bigcup_i K_i$, 3. Each $K_i \cap K_j = \phi$, or $K_i \cap K_j$ is a common edge : 

, or a common vertex : 

4. Each $K_i \subset \Omega$

Let $\{v_i\}_{i=1,\dots,N_v}$ denote the vertices of this triangulation, eg



$$\{v_i\}_{i=1,\dots,N} = \bigcup_i \{T_i(0,0),\ T_i(1,0),\ T_i(0,1)\}$$

Then the Lagrange nodal basis for P^1 associated to the vertices $\{v_i\}_{i=1,\dots,N_v}$ is the collection of functions $\{\phi_i\}_{i=1,\dots,N_v}$ where

1. Each $\phi_i : \Omega \to \mathbb{R}$ is continuous
2. Each $\phi_i \circ T_j : \hat{K} \to \mathbb{R}$ is a degree ≤ 1 polynomial on $\hat{K}$ (for all i,j)
3. Each $\phi_i(v_j) = \delta_{ij}$ for all i,j

d. $-\Delta u = f$ on Ω

$u = 0$ on $\partial\Omega$

weak form: $a(u,\phi) = \int_\Omega \nabla u \cdot \nabla \phi$, $F(\phi) = \int_\Omega f\phi$

$V = H_0^1(\Omega)$

Find $u \in V$ st $a(u,\phi) = F(\phi) \quad \forall \ \phi \in V$.

T_h triangulation of Ω as above.

$$V_h = \underbrace{\{ \phi_h \in C^0(\Omega) \mid \phi_h \circ T_i \in P^1(\hat{K}) \ \forall i \}} \cap \{ \phi : \phi|_{\partial\Omega} = 0 \}$$

Basis $= \{ \phi_i \}_{i=1,\dots,N_v}$ Lagrange nodal basis as above

Discrete problem: Find $u_h = \sum_i \underset{\in \mathbb{R}}{U_i} \phi_i$ st

$$\sum_{i=1}^{N_v} \underbrace{a(\phi_i, \phi_j)}_{A_{ji}} U_i = \underbrace{F(\phi_j)}_{F_j} \quad \text{for all } j = 1,\dots,N_v \text{ st } v_j \notin \partial\Omega$$

A_{ji} stiffness matrix

F_j load vector

$U_i = 0 \quad \forall \ i$ st $v_i = 0$ on $\partial\Omega$

Each element K_i has vertices $v_{i1}, v_{i2}, v_{i3} \in \{v_i\}_{i=1,\dots,N_v}$

This defines a local-to-global enumeration

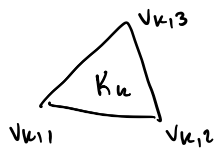
$$I : \underbrace{\{1,\dots,N\}}_{\#\text{ elements}} \times \{1,2,3\} \to \underbrace{\{1,\dots,N_v\}}_{\#\text{ vertices}}.$$

Then $A = \sum_{k=1}^{N} \underbrace{A^k}_{\text{element matrix}}$

$$A^k_{ij} = \int_{K_k} \nabla\phi_i \cdot \nabla\phi_j = 0 \quad \text{if} \quad v_i \notin \{v_{k,1}, v_{k,2}, v_{k,3}\}$$

or

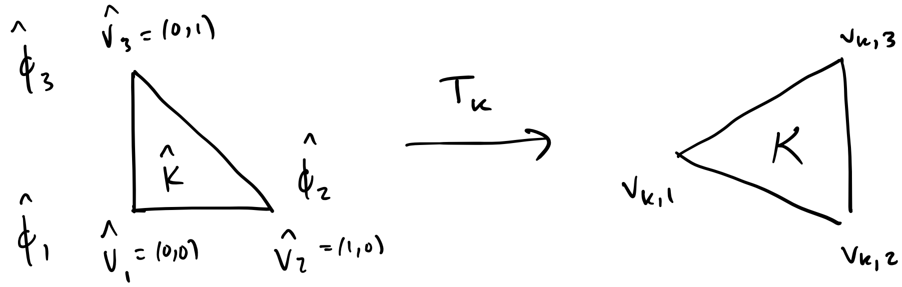
$$v_j \notin \{v_{k,1}, v_{k,2}, v_{k,3}\}$$



when $v_i, v_j \in \{v_{k,1}, v_{k,2}, v_{k,3}\}$ ie $i,j \in \{I(k,1), I(k,2), I(k,3)\}$

$$A^k_{I(k,l), I(k,m)} = \int_{K_k} \nabla\phi_{I(k,l)} \cdot \nabla\phi_{I(k,m)}$$

The vertices are locally enumerated in such a way so that



1. $T_k(\hat{v}_i) = v_{k,i}$,

2. $\phi_{I(k,i)} \circ T_k = \hat{\phi}_i$, where

$$\hat\phi_i : \hat K \to \mathbb{R} \qquad \hat\phi_1(\hat x_1, \hat x_2) = 1 - \hat x_1 - \hat x_2 \, , \quad \nabla \hat\phi_1 = (-1,1)$$

$$\hat\phi_2(\hat x_1, \hat x_2) = \hat x_1 \, , \quad \nabla \hat\phi_2 = (1,0)$$

$$\hat\phi_3(\hat x_1, \hat x_2) = \hat x_2 \qquad \nabla \hat\phi_3 = (0,1)$$

3. $$T_K(\hat x) = v_{K,1} + \underbrace{\begin{bmatrix} v_{K,2} - v_{K,1} & v_{K,3} - v_{K,2} \end{bmatrix}}_{M_K} \hat x$$

4. $$JT_K = M_K \; ; \quad \nabla \hat\phi_i = \nabla \left(\phi_{I(K,i)} \circ T_K \right)$$

$$= \left\{ \left(\nabla \phi_{I(K,i)} \right) \circ T_K \right\} JT_K$$

$$= \left(\nabla \phi_{I(K,i)} \circ T_K \right) M_K$$

$$\longrightarrow \quad \left(\nabla \hat\phi_i \right) M_K^{-1} = \nabla \phi_{I(K,i)} \circ T_K$$

Thus $$A^K_{I(K,l)\, I(K,m)} = \int_{K_K} \nabla \phi_{I(K,l)} \cdot \nabla \phi_{I(K,m)} \, dx$$

$$\xrightarrow{x = T_K(\hat x)} \quad = \int_{\hat K} \left(\nabla \phi_{I(K,l)} \circ T_K \right) \cdot \left(\nabla \phi_{I(K,m)} \circ T_K \right) |\det JT_K| \, d\hat x$$

$$= \int_{\hat K} \underbrace{\left(\nabla \hat\phi_l M_K^{-1} \right) \cdot \left(\nabla \hat\phi_m M_K^{-1} \right) |\det M_K|}_{\text{constant}} \, d\hat x$$

$$\overset{\int_{\hat{K}} d\hat{x} = 1/2}{=} \quad (\nabla \hat{\phi}_\ell M_k^{-1}) \cdot (\nabla \hat{\phi}_m M_k^{-1}) \frac{|\det M_k|}{2}$$

$\therefore$

$$A^k_{ij} = \begin{cases} 0, & i \notin (I(k,1), I(k,2), I(k,3)) \text{ or } j \notin (I(k,1), I(k,2), I(k,3)) \\ (\nabla \hat{\phi}_\ell M_k^{-1}) \cdot (\nabla \hat{\phi}_m M_k^{-1}) \frac{|\det M_k|}{2}, & \text{if } i = I(k,\ell),\ j = I(k,m),\ \ell, m \in \{1,2,3\} \end{cases}$$

and $A = \sum_{k=1}^{N} A^k$.

Similarly, $F = \sum_{k=1}^{N} \underbrace{F^k}_{\text{element load vector}}$

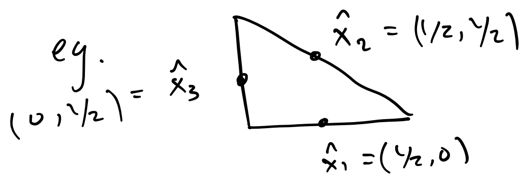
$$F^k_j = \int_{K_k} f \phi_j = \begin{cases} 0, & j \notin \{I(k,1), I(k,2), I(k,3)\} \\ \int_{K_k} f \phi_{I(k,\ell)}, & j = I(k,\ell),\ \ell \in \{1,2,3\}. \end{cases}$$

$$\int_{K_k} f \phi_{I(k,\ell)} \, dx = \int_{\hat{K}} f \circ T_k \, \phi_{I(k,\ell)} \circ T_k \underbrace{|\det M_k|}_{\text{constant}} d\hat{x}$$

$$= |\det M_k| \int_{\hat{K}} f \circ T_k \; \hat{\phi}_\ell \, d\hat{x}$$

$$\approx |\det M_k| \underbrace{\frac{1}{2} \sum_{n=1}^{N_q} w_n \, f(T_k(\hat{x}_n)) \, \hat{\phi}_\ell(\hat{x}_n)}_{\text{quadrature rule on } \hat{K}}$$

eg.



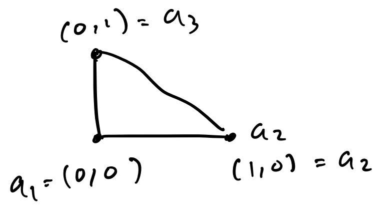
$w_1 = w_2 = w_3 = 1/3$

so $$F = \sum_{k=1}^{N} F^k$$

$$F^k_j = \begin{cases} 0, & j \notin \{I(k,1), I(k,2), I(k,3)\} \\ \frac{|\det M_k|}{2} \sum_{n=1}^{N_q} f(T_k(\hat{x}_n)) \, \hat{\phi}_\ell(\hat{x}_n) + \text{quadrature error} & \text{if } j = I(k,\ell), \\ & \ell \in \{1,2,3\} \end{cases}$$

Post-process Boundary conditions in A, F and then solve for U

e. $P_2(T) = \text{span}\{1, x, y, x^2, xy, y^2\}$

$T =$ 

Need 6 unisolvent dof's

idea: start w/ P^1 dofs : $p(a_i)$ (3 dofs)

Now try to prove unisolvence and add extra dofs as needed.

$p(x,y) = c_1 + c_2 x + c_3 y + c_4 x^2 + c_5 xy + c_6 y^2$

$p(0,0) = c_1 = 0 \rightarrow p(x,y) = c_2 x + c_3 y + c_4 x^2 + c_5 xy + c_6 y^2$

$\rightarrow p(1,0) = c_2 + c_4 = 0 \rightarrow c_2 = -c_4$

$\rightarrow p(x,y) = c_2 x(1-x) + c_3 y + c_5 xy + c_6 y^2 \rightarrow$

$p(0,1) = c_3 + c_6 = 0 \rightarrow c_3 = -c_6 \rightarrow$

$p(x,y) = c_2 x(1-x) + c_3 y(1-y) + c_5 xy$

Now pick 3 other points for dofs. Let's try one of the form $a_4 = (x,0) \rightarrow p(x,0) = c_2 x(1-x)$

choose $0 < x < 1$ so $x, 1-x \neq 0$. Natural choice is $x = 1/2$,

so $a_4 = (1/2, 0)$ and dof #4 is $p(1/2, 0) = \frac{c_2}{4} = 0 \rightarrow$

$c_2 = 0 \rightarrow p(x,y) = c_3 y(1-y) + c_5 xy.$

Symmetry $\rightarrow$ $a_5 = (0, 1/2)$ $\rightarrow$ dof #5 is $p(0, 1/2)$

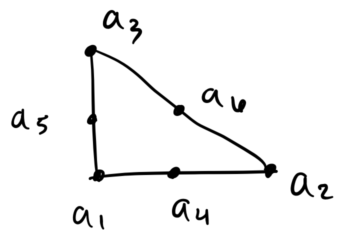
$= \frac{c_3}{4} = 0 \longrightarrow c_3 = 0 \longrightarrow p(x,y) = c_5 xy$

pick dof #6 to be any $(x,y) \in \hat{K}$ st $xy \neq 0$.

Natural choice is $x = y = 1/2$ so dof #6

is $p(1/2, 1/2) = \frac{c_5}{4} = 0 \rightarrow c_5 = 0$

$\therefore$ w/ dof's



If $p(a_i) = 0 \;\; \forall i$ then $p = 0$ $\therefore$ unisolvent.

$P_3(T) = \text{span}\{1, x, y, x^2, xy, y^2, x^3, x^2y, xy^2, y^3\}$

Need 10 DOF's. Trying the same idea as before is messy.

Let's do something else.

Start trying to prove unisolvence and define DOF's along the way:

$$p(x,y) = c_1 + c_2 x + c_3 y + c_4 x^2 + c_5 xy + c_6 y^2 + c_7 x^3 + c_8 x^2 y + c_9 xy^2 + c_{10} y^3$$

Dof #1 $p(0,0) = c_1 = 0$ ✓

Dof #2 $\partial_x p(x,y) = c_2 + 2c_4 x + c_5 y + 3c_7 x^2$

$+ 2c_8 xy + c_9 y^2$

$\partial_x p(0,0) = c_2 = 0$ ✓

$\partial_y p(x,y) = c_3 + c_5 x + 2c_6 y + c_8 x^2 + 2c_9 xy$

$+ 3c_{10} y^2$

Dof #3 $\partial_y p(0,0) = c_3 = 0$ ✓

$p(x,y) = c_4 x^2 + c_5 xy + c_6 y^2 + c_7 x^3 + c_8 x^2 y + c_9 xy^2 + c_{10} y^3$

$p(x,0) = c_4 x^2 + c_7 x^3$

Dof #4 $p(1,0) = c_4 + c_7 = 0 \longrightarrow c_4 = -c_7$

$p(x,y) = c_4 x^2(1-x) + c_5 xy + c_8 x^2 y + c_9 xy^2 + c_{10} y^3$

$+ c_6 y^2$

Dof #5 $p(0,1) = c_6 + c_{10} = 0 \longrightarrow c_6 = -c_{10} \longrightarrow$

$p(x,y) = c_4 x^2(1-x) + c_5 xy + c_6 y^2(1-y) +$

$c_8 x^2 y + c_9 xy^2$

$\partial_x p(x,y) = c_4(2x - 3x^2) + c_5 y + 2c_8 xy + c_9 y^2$

DoF #6 $\quad \partial_x p(1,0) = -c_4 = 0 \rightarrow c_4 = 0$

DoF #7 $\quad \partial_y p(0,1) = -c_6 = 0 \rightarrow c_6 = 0$

$\therefore \quad p(x,y) = c_5 xy + c_8 x^2 y + c_9 xy^2$

$\partial_x p(x,y) = c_5 y + 2c_8 xy + c_9 y^2$

$\partial_y p(x,y) = c_5 x + c_8 x^2 + 2c_9 xy$

DoF #8 $\quad \partial_x p(0,1) = c_5 + c_9 = 0 \rightarrow c_5 = -c_9$

DoF #9 $\quad \partial_y p(1,0) = c_5 + c_8 = 0 \rightarrow c_5 = -c_8$

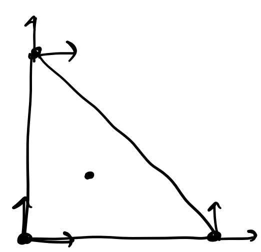
$\Big\} \rightarrow c_8 = c_9$

$\therefore \quad p(x,y) = c_5(xy - x^2y - xy^2)$

DoF #10 $\quad p(1/3, 1/3) = c_5\left(\frac{1}{9} - \frac{1}{27} - \frac{1}{27}\right) = 0 \rightarrow$

$c_5 = 0$

unisolvent!



$P_4(T) = \text{span}\{1, x, y, x^2, xy, y^2, x^3, x^2y, xy^2, y^3,$

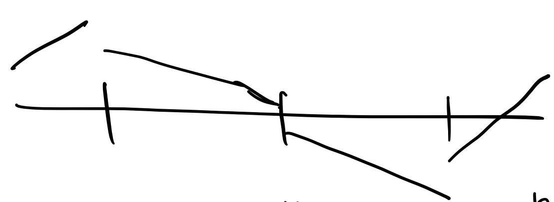
$x^4, x^3y, x^2y^2, xy^3, y^4\}$

Need 15 DOFS for this one. I'm too lazy to do it.

f. Non-conforming : "Broken" finite element space

$V_h^b = \{\phi_h \in L^2(\Omega) \mid \phi_h \circ T_K \in P^1(\hat{K}) \ \forall K \in \mathcal{T}_h\}$

Notice we do <u>not</u> require continuity. In 1D, a typical ϕ_h looks like



Notice the jumps, hence "broken".

the "non-conforming" property

Basic theory tells us $V_h^b \not\subset H^1(\Omega)$

H^1- conforming : Continuous FE space

$V_h^c = \{\phi_h \in C^0(\Omega) \mid \phi_h \circ T_K \in P^1(\hat{K}) \ \forall K\}$

Now we impose continuity

Basic theory tells us $V_h^c \subset H^1(\Omega)$

the conforming property

H^2-conforming : For H^1-conforming, we needed functions to be cts on edges of mesh. For H^2-conforming, we need functions to be C^1 on edges of the mesh. I haven't worked w/ H^2-conforming FE before, but it looks like the FE in III is H^2-conforming.

g. Let V be a Banach space and $a: V \times V \to \mathbb{R}$, $F: V \to \mathbb{R}$ satisfy

1. a is cts, elliptic, bilinear
2. F is cts, linear

Let V_h be a subspace of V. Then we know $\exists$ unique $u \in V$ st

$$a(u, \phi) = F(\phi) \quad \forall \ \phi \in V \quad \text{and}$$

$\exists$ unique $u_h \in V_h$ st $\quad a(u_h, \phi_h) = F(\phi_h) \ \forall \phi_h \in V_h.$

Céa's lemma states that $\exists \ C > 0$ st

$$\|u - u_h\|_V \leq C \inf_{\phi_h \in V_h} \|u - \phi_h\|_V.$$

This estimate is quasi-optimal b/c of the constant C. To see when we obtain the optimal estimate

$$\|u - u_h\|_V = \min_{\phi_h \in V_h} \|u - \phi_h\|_V \quad , \quad \text{Let's}$$

prove Céa's Lemma:

pf. observe: $\quad a(u - u_h, \phi_h) = 0 \quad \forall \phi_h \in V_h$.

Thus $\alpha \|u - u_h\|^2 \overset{\text{coercivity}}{\leq} a(u - u_h, u - u_h)$

$$= a(u-u_h, u-\phi_h)$$

Continuity $\longrightarrow$ $\leq C \|u-u_h\| \|u-\phi_h\|$

$$\rightarrow \quad \|u-u_h\| \leq \frac{C}{\alpha} \|u-\phi_h\| \quad \forall \phi_h \in V_h \qquad \square$$

We observe that if $\boxed{\frac{C}{\alpha} \leq 1}$ ⊛, then the last line of the proof actually reads

$$\|u-u_h\| \leq \|u-\phi_h\| \ \forall \phi_h \in V_h \longrightarrow \|u-u_h\| = \min_{\phi_h \in V_h} \|u-\phi_h\|,$$

so optimality is obtained when ⊛ holds.

One way for this to hold is when

a is $\underbrace{\text{symmetric, cts, coercive, bilinear}}_{\text{this implies } a(\cdot,\cdot) \text{ is an inner product on } V}$

and we take $\|u\|_V = \sqrt{a(u,u)}$ the energy norm.

h. Galerkin orthogonality:

Let $\underbrace{a: V \times V \to \mathbb{R}}_{\text{bilinear}}$, $\underbrace{F: V \to \mathbb{R}}_{\text{linear}}$

and let $V_h \subset V$. Then if there exists $u \in V$ st $a(u, \phi) = F(\phi)$ $\forall \phi \in V$,

and if $\exists\, u_h \in V_h$ st $a(u_h, \phi_h) = F(\phi_h)$ $\forall \phi_h \in V_h$,

we have <u>Galerkin orthogonality</u>:

$$a(u - u_h, \phi_h) = 0 \quad \forall \phi_h \in V_h$$

pf.

$$a(u, \phi_h) \underset{V_h \subset V}{=} F(\phi_h) = a(u_h, \phi_h) \rightarrow$$

$$a(u - u_h, \phi_h) \overset{\text{bilinear}}{=} a(u, \phi_h) - a(u_h, \phi_h)$$

$$= F(\phi_h) - F(\phi_h) = 0$$

$\square$

i, j : These should be in the class notes.

III. SKIP!

IV. Covered in recitation + some results are in the notes

V. Check your old HW's / my old HW Hints