**Definition** Let $K$ be a polygon (square or triangle), $P(K)$ a space of polynomials defined on $K$, and $\Sigma$ a set of linear functionals on $P(K)$, called degrees of freedom.
Then the triple $(K, P(K), \Sigma)$ is **unisolvent** if

1. $\# \Sigma = \dim P(K)$
2. For any $p \in P(K)$, if $\sigma(p) = 0$ for all $\sigma \in \Sigma$, then $p = 0$.

**Remarks** • A member of $P(K)$ is a polynomial $p : K \to \mathbb{R}$. For example, if $K$ is the unit square and $P(K)$ is the space of all degree at most 2 polynomials on $K$, then a typical member of $P(K)$ is of the form $p(x,y) = c_1 + c_2 x + c_3 y + c_4 x^2 + c_5 xy + c_6 y^2$ for some constants $c_1, c_2, c_3, c_4, c_5, c_6$

- A member of $\Sigma$ is a linear map $\sigma : P(K) \to \mathbb{R}$. That is, $\sigma$ assigns to each polynomial $p \in P(K)$ a real number $\sigma(p) \in \mathbb{R}$, and for any scalar $c \in \mathbb{R}$, any 2 polynomonials $p_1, p_2 \in P(K)$

$$\sigma(c p_1 + p_2) = c\,\sigma(p_1) + \sigma(p_2).$$

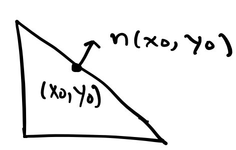
For example, if we fix a point $(x_0, y_0) \in K$, then $\sigma(p) = p(x_0, y_0)$ is a valid degree of freedom.

Other examples include $\sigma(p) = \partial_x p(x_0, y_0)$,

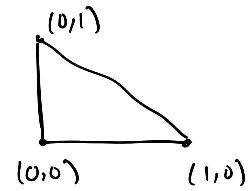
$$\sigma(p) = \int_K p(x,y)\,dx\,dy,$$

$$\sigma(p) = \underbrace{\partial_n p(x_0, y_0)}_{\text{normal derivative}} = \nabla p(x_0, y_0) \cdot \underbrace{n(x_0, y_0)}_{\text{outward unit normal}}$$

for $(x_0, y_0)$ on boundary of $K$ eg



- Condition 1 of unisolvence means that the dimension of the vector space $P(K)$ must be the same as the number of degrees of freedom (DOF's)

For example, w/ $K =$ 


$$P(K) = \underbrace{P_1(K)}_{\text{deg} \leq 1 \text{ polynomials on } K} = \text{span}\{1, x, y\}$$

$$\sigma_1(p) = p(0,0) \qquad \sigma_i : P(K) \to \mathbb{R}$$

$$\sigma_2(p) = p(1,0) \qquad \Sigma = \{\sigma_1, \sigma_2, \sigma_3\}$$

$$\sigma_3(p) = p(0,1)$$

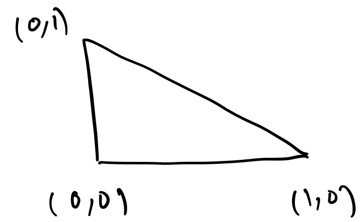
we have $\dim P(K) = 3 = \# \Sigma$

On the other hand, if we keep $\Sigma$ the same but now take $P(K) = \underbrace{P_2(K)}_{\text{deg} \leq 2 \text{ polynomials}} = \text{span}\{1, x, y, x^2, xy, y^2\}$

we have $\dim P(K) = 6 \neq \# \Sigma = 3$.

- Condition 2 of unisolvence says that the *only* polynomial that can vanish on / annihilate all of the DOF's is the zero polynomial.

For example,

$K =$ 

$P(K) = P_1(K)$

$\sigma_1(p) = p(0,0) \qquad \sigma_2(p) = p(1,0) \qquad \sigma_3(p) = p(0,1)$

$\Sigma = \{\sigma_1, \sigma_2, \sigma_3\}$

claim: $(K, P(K), \Sigma)$ is unisolvent.

proof. $\dim P(K) = 3 = \# \Sigma$

Now let $p \in P(K)$ satisfy

$\sigma_1(p) = p(0,0) = 0, \qquad \sigma_3(p) = p(0,1) = 0.$

$\sigma_2(p) = p(1,0) = 0,$

We will show this implies $p = 0$.

Since $p \in P(K) = \text{span}\{1, x, y\}$,

$\exists$ constants $c_1, c_2, c_3 \in \mathbb{R}$ so

$$p(x,y) = c_1 + c_2 x + c_3 y \quad \forall (x,y) \in K$$

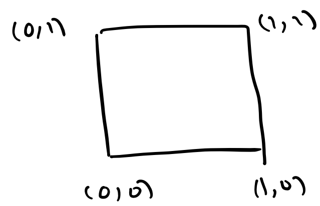
Then
$$\left.\begin{aligned} p(0,0) &= c_1 = 0 \\ p(1,0) &= c_1 + c_2 = 0 \\ p(0,1) &= c_1 + c_3 = 0 \end{aligned}\right\} \rightarrow c_1 = c_2 = c_3 = 0$$

$\rightarrow \; p(x,y) = 0 \quad \forall x, y \;\rightarrow\; p = 0.$

Thus conditions 1,2 are satisfied, so

$(K, P(K), \Sigma)$ is unisolvent. $\square$

<u>Another example</u>

$K =$ 


$P(K) = P_2(K) = \text{span}\{1, x, y, x^2, y^2, xy\}$

$\sigma_1(p) = p(0,0)$ $\qquad$ $\sigma_4(p) = p(1,1)$

$\sigma_2(p) = p(1,0)$ $\qquad$ $\sigma_5(p) = p(1/2, 1/2)$

$\sigma_3(p) = p(0,1)$ $\qquad$ $\sigma_6(p) = \int_K p(x,y)\,dx\,dy$

claim : $(K, P(K), \Sigma)$ is <u>not</u> unisolvent.

proof. $\dim P(K) = 6 = \# \Sigma$. Now suppose $p \in P(K)$ satisfies $\sigma_i(p) = 0 \ \forall i$.

Then $p \in P(K) \to \exists\, c_1, c_2, c_3, c_4, c_5, c_6 \in \mathbb{R}$ so

$$p(x,y) = c_1 + c_2 x + c_3 y + c_4 x^2 + c_5 xy + c_6 y^2 \quad \forall (x,y) \in K$$

$\therefore$

$\sigma_1(p) = p(0,0) = c_1 = 0$ $\quad$ (1)

$\sigma_2(p) = p(1,0) = c_1 + c_2 + c_4 = 0$ $\quad$ (2)

$\sigma_3(p) = p(0,1) = c_1 + c_3 + c_6 = 0$ $\quad$ (3)

$\sigma_4(p) = p(1,1) = c_1 + c_2 + c_3 + c_4 + c_5 + c_6 = 0$ $\quad$ (4)

$\sigma_5(p) = p(1/2, 1/2) = c_1 + c_2/2 + c_3/2 + c_4/4 + c_5/4 + c_6/4 = 0$ $\quad$ (5)

$$\sigma_6(p) = \int_K p(x,y)\,dx\,dy = \int_0^1\int_0^1 c_1 + c_2x + c_3y + c_4x^2 + c_5xy + c_6y^2\,dx\,dy$$

$$= c_1 + \frac{c_2}{2} + \frac{c_3}{2} + \frac{c_4}{3} + \frac{c_5}{4} + \frac{c_6}{3} = 0 \quad (6)$$

Equations (1) – (6) can be put in matrix-vector form like

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1/2 & 1/2 & 1/4 & 1/4 & 1/4 \\ 1 & 1/2 & 1/2 & 1/3 & 1/4 & 1/3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \end{bmatrix}}_{\vec{c}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

If $\det A \neq 0$, then $A^{-1}$ exists, so

$$A\vec{c} = 0 \rightarrow A^{-1}A\vec{c} = A^{-1}0 \rightarrow \vec{c} = 0 \rightarrow p = 0$$

$\therefore$ unisolvent!

If $\det A = 0$ then $A^{-1}$ does not exist,

and there must exist some choice of

$c_1, \ldots, c_6 \in \mathbb{R}$ w/ some $c_i \neq 0$ so

$$Ac = 0$$

ie $p(x,y) = c_1 + c_2 x + c_3 y + c_4 x^2 + c_5 xy + c_6 y^2$

is not the zero polynomial but

all $\sigma_i(p) = 0$. Thus condition #2 fails

to hold, so $(K, P, \Sigma)$ is not

unisolvent.

Thus $(K, P, \Sigma)$ is unisolvent $\Leftrightarrow$ $\det A \neq 0$.

Use a computer (or do it by hand) to

compute $\det A = 0$, so $(K, P, \Sigma)$ is

not unisolvent.

Indeed $p(x,y) = x - x^2 - y + y^2$ satisfies

$p(0,0) = 0$ $\quad p(1/2, 1/2) = 0$

$p(1,0) = 0$ $\quad \int_0^1 \int_0^1 x - x^2 - y + y^2 \, dx\, dy = 0$

$p(0,1) = 0$

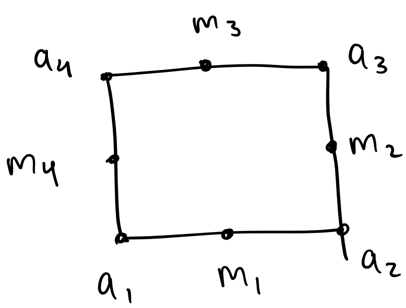
$p(1,1) = 0$ $\quad$ yet $p \neq 0$.

$\square$

## DOFS for HW problems

HW 6 Q III

$\sigma_i(p) = p(m_i)$

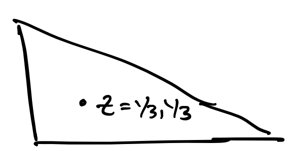



$\sigma_i(p) = p(a_i)$

by $\nabla p(a_i)$ we mean $\sigma_i^1(p) = \partial_x p(a_i)$

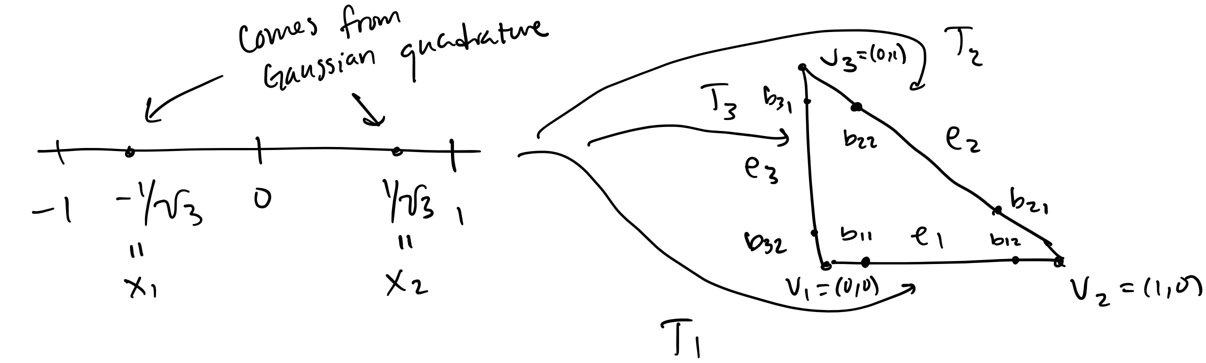
$\sigma_i^2(p) = \partial_y p(a_i)$

HW 7 Q II $\quad K =$




$\sigma_i(p) = p(z)$

$b_{ij}$ are the Gauss points:




$b_{ij} = T_i(x_j)$ where

$$T_1(x) = V_1\left(\frac{1-x}{2}\right) + V_2\left(\frac{1+x}{2}\right)$$

$$T_2(x) = V_2\left(\frac{1-x}{2}\right) + V_2\left(\frac{1+x}{2}\right)$$

$$T_3(x) = V_3\left(\frac{1-x}{2}\right) + V_1\left(\frac{1+x}{2}\right)$$

$T_i : [-1,1] \to e_i$ edge of $K$

by $\nabla^2 p(a_i)$ we mean

$$\left.\begin{aligned} \sigma_i^1 &= \partial x^2\, p(a_i) \\ \sigma_i^2 &= \partial y^2\, p(a_i) \\ \sigma_i^3 &= \partial x \partial y\, p(a_i) \end{aligned}\right\} \text{2nd-order derivatives}$$

by $\partial_n p(m_i)$ we mean

dot product

$$\sigma_i(p) = \partial_n p(m_i) = \nabla p(m_i) \cdot n(m_i)$$

normal derivative at $m_i$

where