MATH 610 Homework 1 Hints

Jordan Hoffart

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1 Exercise 1

1.1 Problem 1

A function u belongs to $H^1(-1,1)$ if and only if

1. u belongs to $L^2(-1,1)$,
2. u has a weak derivative in $L^2(-1,1)$.

A function u belongs to $L^2(-1,1)$ if and only if the integral

$$\int_{-1}^{1} u(x)^2 \, \mathrm{d}x$$

exists and is finite. One way to show this is to explicitly compute the integral. There is a more elegant way to do this without computing anything. I'll let you figure that one out.

To find a weak derivative of u, let φ be a test function, meaning that $\varphi \in C_0^\infty([-1,1])$, which means that

1. φ is infinitely differentiable,
2. $\varphi(-1) = 0$,
3. $\varphi(1) = 0$.

Then split up the integral over the pieces of u

$$\int_{-1}^{1} u(x)\varphi'(x) \, \mathrm{d}x = \int_{-1}^{0} u(x)\varphi'(x) \, \mathrm{d}x + \int_{0}^{1} u(x)\varphi'(x) \, \mathrm{d}x$$

and then do integration by parts. See what falls out at the end to find a candidate for the weak derivative v of u, and then check if $v \in L^2(-1,1)$.

1.2 Problem 2

This is a generalization of problem 1, so we proceed similarly. This time, explicitly compute the integral of $u(x)^2$ where $u(x) = |x|^\alpha$ and see which values of α give you a finite integral. This tells you which α allows $u \in L^2(a,b)$. Then let φ be a test function, do integration by parts over the pieces of u as in problem 1, and see what falls out to give you a candidate for the weak v derivative of u. This will tell you which α allows for the integration-by-parts to happen at all and thus give you a candidate. Then compute the integral of $v(x)^2$ to see which α allow for $v \in L^2(-1,1)$.

1.3 Problem 3

Be careful here. Observe that if φ is a test function on $[-1,1]$, this does not mean that the restriction $\varphi|_{[-1,0]}$ is a test function on $[-1,0]$, nor is it a test function when restricted to $[0,1]$. Therefore, we cannot immediately do the following calculation

$$\begin{aligned}\int_{-1}^{1} u(x)\varphi'(x)\,\mathrm{d}x &= \int_{-1}^{0} u_1(x)\varphi'(x)\,\mathrm{d}x + \int_{0}^{1} u_2(x)\varphi'(x)\,\mathrm{d}x \\ &= -\int_{-1}^{0} u_1'(x)\varphi(x)\,\mathrm{d}x - \int_{0}^{1} u_2'(x)\varphi(x)\,\mathrm{d}x\end{aligned}$$

since we can only go to the second line when $\varphi|_{[-1,0]}$ is a test function on $[-1,0]$ and $\varphi|_{[0,1]}$ is a test function on $[0,1]$. We have to be a bit more clever here to justify this. To start, we need the following theorem.

Theorem 1. *If $u \in H^1(a,b)$, then there is a sequence u_n of smooth functions in $C^\infty(a,b)$ that converges to u in $H^1(a,b)$.*

We apply this to u_1 and u_2 to get a sequence u_n^1 in $H^1(-1,0)$ and a sequence u_n^2 in $H^1(0,1)$ where u_n^i converges to u_i. Let

$$u_n(x) = \begin{cases} u_n^1(x) & x \in (-1,0) \\ u_n^2(x) & x \in [0,1) \end{cases}$$

Then we have that

$$\begin{aligned}\int_{-1}^{1} u_n(x)\varphi'(x)\,\mathrm{d}x &= \int_{-1}^{0} u_n^1(x)\varphi'(x)\,\mathrm{d}x + \int_{0}^{1} u_n^2(x)\varphi'(x)\,\mathrm{d}x \\ &= -\int_{-1}^{0} (u_n^1)'(x)\varphi(x)\,\mathrm{d}x - \int_{0}^{1} (u_n^2)'(x)\varphi(x)\,\mathrm{d}x + (u_n^1(0) - u_n^2(0))\varphi(0) \\ &= -\int_{-1}^{0} v_n(x)\varphi(x)\,\mathrm{d}x + (u_n^1(0) - u_n^2(0))\varphi(0)\end{aligned}$$

where the second step follows from integration-by-parts, which is allowed now because the u_n^i are classically smooth and not only in H^1, and

$$v_n(x) = \begin{cases} (u_n^1)'(x) & x \in (-1, 0) \\ (u_n^2)'(x) & x \in [0, 1) \end{cases}.$$

Set

$$v(x) = \begin{cases} u_1'(x) & x \in (-1, 0) \\ u_2'(x) & x \in [0, 1) \end{cases}.$$

Show that

$$\begin{aligned} \int_{-1}^{1} u_n(x)\varphi'(x)\,\mathrm{d}x &\to \int_{-1}^{1} u(x)\varphi'(x)\,\mathrm{d}x, \\ \int_{-1}^{1} v_n(x)\varphi(x)\,\mathrm{d}x &\to \int_{-1}^{1} v(x)\varphi(x)\,\mathrm{d}x, \\ (u_n^1(0) - u_n^2(0))\varphi(0) &\to 0, \\ v &\in L^2(-1, 1) \end{aligned}$$

as $n \to \infty$ and conclude that $u \in H^1(-1, 1)$ with v as its weak derivative. You are free to use the following theorem, which is a consequence of the optional problem 5.

Theorem 2. *Fix $x_0 \in [a, b]$. Then the map*

$$E_{x_0}(u) = u(x_0)$$

is a continuous linear functional on $H^1(a, b)$. In other words,

$$E_{x_0}(cu + v) = cE_{x_0}(u) + E_{x_0}(v)$$

for all $u, v \in H^1(a, b)$ and all $c \in \mathbb{R}$, and there is a constant $C > 0$ such that

$$|E_{x_0}(u)| \le C\|u\|_{H^1(a,b)}$$

for all $u \in H^1(a, b)$.

1.4 Problem 4

Let $u \in H^1(a, b)$. Then there is a sequence v_n of smooth functions in $C^\infty(a, b)$ that converges to u in $H^1(a, b)$. Use the fact that

$$\|v\|_{L^\infty(a,b)} \le C\|v\|_{H^1(a,b)}$$

when $v \in C^\infty(a, b)$ (which is a consequence of the last inequality in question 2 of exercise 2) to argue that v_n is Cauchy in $L^\infty(a, b)$. Since $L^\infty(a, b)$ is complete, this implies that there exists $v \in L^\infty(a, b)$ such that $v_n \to v$ in $L^\infty(a, b)$. Now argue that $u = v$.

2 Exercise 2

2.1 Problem 1

Use the triangle inequality and the Cauchy-Schwarz inequality

$$\begin{aligned}|u(x)| &\le |u(y)| + \int_0^1 |u'(s)|\,\mathrm{d}s \\ &= |u(y)| + (1, |u'|)_{L^2(0,1)} \\ &\le |u(y)| + \|u'\|_{L^2(0,1)}.\end{aligned}$$

Now pick particular points for y.

2.2 Problem 2

For the first inequality, first integrate

$$u(x) = \int_0^1 u(y)\,\mathrm{d}y + \int_0^1 \int_y^x u'(s)\,\mathrm{d}s\,\mathrm{d}y.$$

Then use the triangle inequality and Cauchy-Schwarz:

$$|u(x)| \le |\overline{u}| + \|u'\|_{L^2(0,1)}$$

where

$$\overline{u} = \int_0^1 u(y)\,\mathrm{d}y.$$

At some point you will need to use Young's inequality:

$$2ab \le a^2 + b^2,$$

which follows from the fact that

$$(a-b)^2 \ge 0$$

for all $a, b \in \mathbb{R}$.

For the second and third inequalities, do essentially the same thing as problem 1, pick particular points for y, and apply Young's inequality.

For the last inequality, proceed as in problem 1 to get

$$|u(x)| \le |u(y)| + \|u'\|_{L^2(0,1)}.$$

Then square, apply Young's inequality, and integrate.