MATH 610 Homework 3 Hints

Jordan Hoffart

July 25, 2025

1 Exercise 1

1.1 Problem 1

Multiply by a test function, integrate by parts, and use the boundary conditions. Find the correct Sobolev space V, the right bilinear form $a : V \times V \to \mathbb{R}$, and the right linear form $F : V \to \mathbb{R}$ such that the variational problem reads as follows: Find $u \in V$ such that

$$a(u, v) = F(v)$$

for all $v \in V$.

1.2 Problem 2

You have to solve problem 1 to get the answer for this problem as well, so the hint is the same.

1.3 Problem 3

First find the basis functions for the unit interval $(0, 1)$. In other words, find $\widehat{\phi}_i$ for $i = 1, 2, 3$ that are quadratic polynomials over $(0, 1)$ and which

$$\begin{aligned}
&\widehat{\phi}_1(0) = 1, && \int_0^1 \widehat{\phi}_1(\widehat{x})\,\mathrm{d}\widehat{x} = 0, && \widehat{\phi}_1(1) = 0,\\
&\widehat{\phi}_2(0) = 0, && \int_0^1 \widehat{\phi}_2(\widehat{x})\,\mathrm{d}\widehat{x} = 1, && \widehat{\phi}_2(1) = 0,\\
&\widehat{\phi}_3(0) = 0, && \int_0^1 \widehat{\phi}_3(\widehat{x})\,\mathrm{d}\widehat{x} = 0, && \widehat{\phi}_3(1) = 1.
\end{aligned}$$

Now we map $(0, 1)$ onto (x_j, x_{j+1}) via

$$T_j(\widehat{x}) = x_j + (x_{j+1} - x_j)\widehat{x}. \tag{1}$$

Convince yourself (and me) that the basis function ϕ_i^j on (x_j, x_{j+1}) that you are looking for is just given by

$$\phi_i^j(x) = \widehat{\phi}_i(T_j^{-1}(x))$$

for all $x \in (x_j, x_{j+1})$.

1.4 Problem 4

The element stiffness matrix S_j and the element mass matrix M_j are given by

$$\begin{aligned}(S_j)_{i,k} &= \int_{x_j}^{x_{j+1}} \frac{\mathrm{d}}{\mathrm{d}x}\phi_i^j(x)\frac{\mathrm{d}}{\mathrm{d}x}\phi_k^j(x)\,\mathrm{d}x,\\ (M_j)_{i,k} &= \int_{x_j}^{x_{j+1}} \phi_i^j(x)\phi_k^j(x)\,\mathrm{d}x.\end{aligned}$$

Use the change of coordinates (1) to transform these integrals into integrals over $(0,1)$ involving the basis functions $\widehat{\phi}_i$ to simplify the computation.

1.5 Problem 5

The homework has a typo in it. We define the space V_h as the space of piecewise quadratics over the splitting (x_j, x_{j+1}) without specifying any kind of continuity. However, the variational problem is posed on a subspace V of $H^1(0,1)$. Since functions in $H^1(0,1)$ are continuous, so are functions in V. Since we are working in the conforming setting, i.e. $V_h \subset V$, we must specify that V_h consist of *continuous* piecewise quadratics on the splitting, otherwise what we are doing doesn't fit into our theoretical framework.

The Ritz system is to find $u_h \in V_h$ such that

$$a(u_h, v_h) = F(v_h)$$

for all $v_h \in V_h$. Since V_h is finite dimensional, we can choose a basis $\psi_1, \ldots, \psi_m$ for V_h and arrive at the equivalent matrix-vector problem of finding the vector $\vec{u}_h$ of coefficients of u_h with respect to the ψ_i such that

$$A_h\vec{u}_h = \vec{F}_h, \tag{2}$$

where

$$\begin{aligned}(A_h)_{i,j} &= a(\psi_j, \psi_i),\\ (\vec{F}_h)_i &= F(\psi_i),\\ u_h &= \sum_{j=1}^{m}(\vec{u}_h)_j\psi_j.\end{aligned} \tag{3}$$

The particular basis that we choose for V_h is constructed from the ϕ_i^j in the following way. First, we observe that $\phi_2^j = 0$ at the endpoints (x_j, x_{j+1}), so we can extend these by zero to be functions in V_h. In other words, we let

$$\psi_{j+1}(x) = \begin{cases} \phi_2^j(x) & x \in (x_j, x_{j+1}) \\ 0 & \text{otherwise} \end{cases}$$

for $j = 0, \dots, n-1$. This gives us n basis functions defiend so far. Next, on two adjacent intervals (x_{j-1}, x_j) and (x_j, x_{j+1}), we have that $\phi_3^{j-1}(x_j) = \phi_1^j(x_j) = 1$, while $\phi_3^{j-1}(x_{j-1}) = 0$ and $\phi_1^j(x_{j+1}) = 0$. Therefore, we may set

$$\psi_{n+j}(x) = \begin{cases} \phi_3^{j-1}(x) & x \in (x_{j-1}, x_j) \\ \phi_1^j(x) & x \in (x_j, x_{j+1}) \\ 0 & \text{otherwise} \end{cases}$$

for $j = 1, \dots, n-1$. This now gives us $n-1$ more basis functions, so we have $2n$ basis functions defined so far. Finally, since $\phi_1^0(x_1) = 0$ and $\phi_3^{n-1}(x_{n-1}) = 0$, we set

$$\psi_{2n}(x) = \begin{cases} \phi_1^0(x) & x \in (x_0, x_1) \\ 0 & \text{otherwise} \end{cases},$$

$$\psi_{2n+1}(x) = \begin{cases} \phi_3^{n-1}(x) & x \in (x_{n-1}, x_n) \\ 0 & \text{otherwise} \end{cases}.$$

This gives us a grand total of $m = 2n + 1$ basis functions.

The global stiffness and mass matrices S and M are then defined as

$$S_{i,j} = \int_0^1 \psi_j'(x)\psi_i'(x)\,\mathrm{d}x,$$

$$M_{i,j} = \int_0^1 \psi_j(x)\psi_i(x)\,\mathrm{d}x.$$

To compute these entries, split up the integrals over the elements (x_j, x_{j+1}), consider which integrals are nonzero, and use the element-wise stiffness and mass matrices from the previous problem.

Remark 1. The ordering of the basis is not unique. Here is a re-ordering of the basis above that can be more convenient for writing down the globally assembled stiffness and mass matrices of the problem.

First, we set $\theta_1 = \psi_{2n}$. Then we set $\theta_2 = \psi_1$. Then we set $\theta_3 = \psi_{n+1}$. Observe that, when restricted to the first subinterval (x_0, x_1), θ_1 corresponds to ϕ_1^0, θ_2 corresponds to ϕ_2^0, and θ_3 corresponds to ϕ_3^0.

We proceed similarly for the next subinterval, setting $\theta_4 = \psi_2$ and $\theta_5 = \psi_{n+2}$. Then θ_3 corresponds to ϕ_1^1, θ_4 corresponds to ϕ_2^1, and θ_5 corresponds to ϕ_3^1 on the subinterval (x_1, x_2).

In general, for interior subintervals (x_j, x_{j+1}) with $1 \leq j \leq n-2$, we have the global basis functions $\theta_{2+3(j-1)+1} = \psi_{n+j}$, $\theta_{2+3(j-1)+2} = \psi_{j+1}$; while for the first subinterval we have $\theta_1 = \psi_{2n+1}$ and $\theta_2 = \psi_1$ and the last subinterval (x_{n-1}, x_n) we have $\theta_{2n-1} = \psi_{2n-1}$, $\theta_{2n} = \psi_n$, and $\theta_{2n+1} = \psi_{2n+1}$.

This ordering of the basis functions is more localized in the sense that basis function θ_j only has nonzero interactions with basis functions θ_{j-1}, itself, and θ_{j+1}. However, it is less convenient to write down than the previous one.

1.6 Problem 6

The right hand side of the Ritz system is just given by (3). If we replace the boundary condition at $x = 0$, then the space of the variational problem V changes as well as the conforming finite element space V_h and the bilinear form a and linear form F. Call the new discrete space V_{h0}, the new bilinear form a_0, and the new linear form F_0. Using the basis of V_h, determine the corresponding basis for V_{h0} constructed as in the last problem, and use this new basis to recompute the Ritz system (2).