Bounding the area of a triangle

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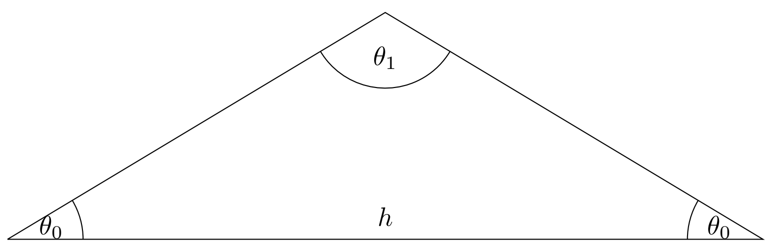
In this note, we answer two questions. Given a triangle K of diameter h with smallest angle θ_0,

1. what is the smallest its area can be, and
2. what is the largest its area can be?

For the first question, consider an isosceles triangle with two angles equal to θ_0.



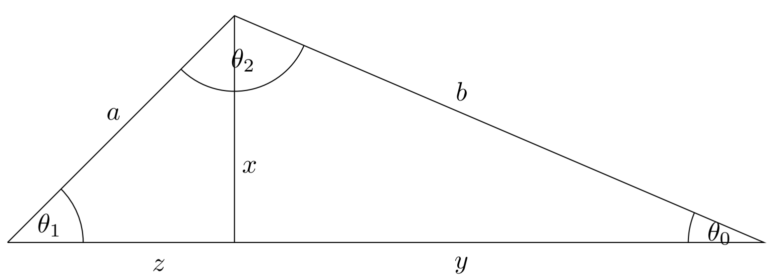
Then the longest side length h must be opposite of the third angle $\theta_1 = \pi - 2\theta_0$.

A simple calculation of its area using the base h and the height $x = (h/2)\tan\theta_0$ gives us the area

$$A_{min} = \frac{h^2}{4}\tan\theta_0.$$

We will show that this is the smallest area that can be obtained by a triangle with such restrictions as above.

Indeed, without loss of generality, such a triangle is of the form below.



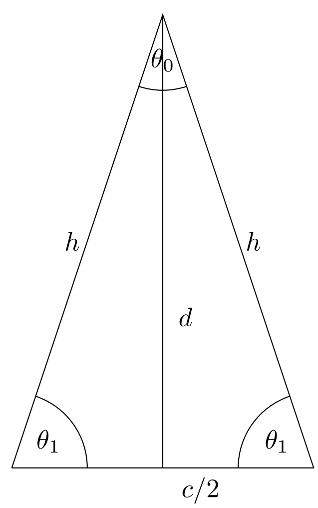
Here, $\theta_0 < \theta_1 < \theta_2$, $y+z=h$, and x is the height of the triangle. Then we have that

$$\begin{aligned} A &= \frac{1}{2}xh \\ &= \frac{1}{2}x(y+z) \\ &= \frac{1}{2}y^2\tan\theta_0 + \frac{1}{2}z^2\tan\theta_1 \\ &\geq \frac{1}{2}(y^2+z^2)\tan\theta_0 \\ &= \frac{1}{2}(y^2+(h-y)^2)\tan\theta_0. \end{aligned}$$

Let $f(y) = y^2 + (h-y)^2$. Then basic calculus tells us that f is minimized at $y = h/2$ with a value of $h^2/2$. Therefore,

$$\begin{aligned} A &= \frac{1}{2}(y^2+(h-y)^2)\tan\theta_0 \\ &\geq \frac{h^2}{4}\tan\theta_0 \\ &= A_{min}. \end{aligned}$$

Now we consider the second question. For this, we consider once again an isosceles triangle, but this time with with two sides equal to h and the third angle equal to θ_0.



In this case,

$$\sin(\theta_0/2) = c/(2h)$$

and

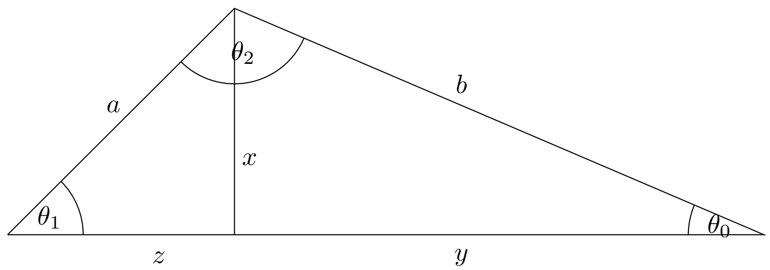
$$\cos(\theta_0/2) = d/h$$

so that the area satisfies

$$A_{max} = \frac{1}{2}cd = h^2 \sin(\theta_0/2)\cos(\theta_0/2) = \frac{h^2}{2}\sin\theta_0,$$

where in the last step we used the double angle identity for sine.

We will now show that this is the largest that the area can be for a general not necessarily isosceles triangle.



Indeed, we have that $b < h$, and therefore

$$A = \frac{1}{2}bh\sin\theta_0 \leq A_{max}.$$

Thus, we get the following theorem.

Theorem 1. *For any triangle K of diameter h and smallest angle θ_0, its area A_K is bounded by*

$$\frac{1}{4}h^2\tan\theta_0 \leq A_K \leq \frac{1}{2}h^2\sin\theta_0,$$

and these bounds are sharp (we cannot improve them).

As a consequence, if we have a triangular mesh where all triangles have angles no smaller than θ_0 and all areas no larger than A_0, we have that the diameter h_K of a triangle in the mesh satisfies

$$A_0 \geq A_K \geq \frac{h_K^2}{4}\tan\theta_0,$$

so that

$$h_K \leq 2\sqrt{A_0\cot\theta_0}$$

for all triangles in the mesh. If we then choose A_0 so that

$$2\sqrt{A_0\cot\theta_0} \leq 1/n$$

when we have that $h_K \leq 1/n$ for all K. This is accomplished if we set

$$A_0 \leq \frac{\tan\theta_0}{4n^2}.$$

Remark 1. Since the smallest angle θ_0 of any triangle must satisfy $\theta_0 \leq \pi/3$ (otherwise the angles would add up to something strictly larger than π), the minimal angle condition is only useful for such θ_0.