Alternating Series Test

Theorem If the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + \cdots$$

with $b_n \geq 0$ satisfies

(1) $b_{n+1} \leq b_n$

(2) $\lim_{n \to \infty} b_n = 0$

then the series is convergent.

Example

The alternating harmonic series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

satisfies (1) $\frac{1}{n+1} < \frac{1}{n}$

(2) $\lim_{n \to \infty} \frac{1}{n} = 0$ so it converges by the alternating series test.

Definition (Absolute Convergence)

A series $\sum a_n$ converges absolutely if $\sum |a_n|$ converges.

Theorem If $\sum a_n$ converges absolutely, then $\sum a_n$ converges.

Example The series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$ converges absolutely since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p-series. Therefore, $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$ also converges.

Example The alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ converges by the alternating series test, but it does <u>not</u> converge absolutely since

$$\sum_{n=1}^{\infty} \frac{1}{n} \quad \text{is a divergent } p\text{-series.}$$

Therefore, convergence does <u>not</u> imply absolute convergence.

<u>Definition</u> (Conditional Convergence)

A series $\sum a_n$ is conditionally convergent if it converges but not absolutely.

<u>Example</u> The alternating harmonic series is conditionally convergent.

<u>Theorem</u> (Ratio Test)

(1) If $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then

$\displaystyle\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

(2) If $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$ (or ∞),

then the series $\sum a_n$ is divergent.

(3) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$, then the ratio test is inconclusive.

Example Consider the series $\sum_{n=1}^{\infty} \frac{(-1)^n n^3}{3^n}$.

Then since $\left| \dfrac{\frac{(-1)^{n+1}(n+1)^3}{3^{n+1}}}{\frac{(-1)^n n^3}{3^n}} \right| = \left(\frac{n+1}{n}\right)^3 \cdot \frac{1}{3}$

which converges to $\frac{1}{3}$ as $n \to \infty$,

we conclude from the ratio test that the series is absolutely convergent.

Exercises

1. Determine if the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+1}$ absolutely converges, conditionally converges, or diverges.

Answer. For absolute convergence, we see if

$$\sum_{n=1}^{\infty} \frac{n^2}{n^3+1}$$ converges.

We can use the limit comparison test with

$$b_n = \frac{1}{n} \quad \text{and} \quad a_n = \frac{n^2}{n^3+1}.$$

Then $$\frac{a_n}{b_n} = \frac{n^2}{n^2+1/n} = \frac{1}{1+1/n^3} \to 1 \text{ as } n \to \infty.$$

Then since $\sum b_n$ diverges, we conclude that the series $\sum (-1)^n \frac{n^2}{n^3+1}$ is <u>not</u> absolutely convergent.

However, it may still converge, so we must check this.

For this, we will use the alternating series test.

We have that $\dfrac{n^2}{n^3+1} = \dfrac{\frac{1}{n}}{1+1/n^3} \to 0$ as $n \to \infty$.

Now we need to verify that $\dfrac{(n+1)^2}{(n+1)^3+1} \leq \dfrac{n^2}{n^3+1}$

(at least for large enough n).

To do this, we let $f(x) = \dfrac{x^2}{x^3+1}$

and we compute $f'(x) = \dfrac{(x^3+1)(2x) - x^2(3x^2)}{(x^3+1)^2}$

$$= \frac{2x - x^4}{(x^3+1)^2} = \frac{x(2-x^3)}{(x^3+1)^2}.$$

For $x > 0$, we see that $f'(x) \leq 0$ if

$2 - x^3 \leq 0 \iff x \geq \sqrt[3]{2}$.

Therefore, for $x \geq 2$, we have that $f(x)$ is decreasing, so that $f(n+1) = \dfrac{(n+1)^2}{(n+1)^3+1} < \dfrac{n^2}{n^3+1} = f(n)$

when $n \geq 2$. Thus, we conclude from the alternating series test that $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+1}$ is conditionally convergent. $\square$

2. Same as problem 1, but now for

$$\sum_{n=1}^{\infty} (-1)^n \frac{n^2}{n^2+n+1}.$$

Answer. Since $\frac{n^2}{n^2+n+1} = \frac{1}{1+\frac{1}{n}+\frac{1}{n^2}} \to 1$

as $n \to \infty$, the test for divergence tells us that $\sum_{n=1}^{\infty} \frac{n^2}{n^2+n+1}$ diverges, so

$\sum_{n=1}^{\infty} (-1)^n \frac{n^2}{n^2+n+1}$ is not absolutely convergent.

It also does not satisfy the assumptions of the alternating series test, so we cannot apply it here. However, since $\frac{n^2}{n^2+n+1} \to 1$, this implies that $(-1)^n \dfrac{n^2}{n^2+n+1}$ does <u>not</u> converge

at all, so the test for divergence once again

tells us that the series $\sum (-1)^n \dfrac{n^2}{n^2+n+1}$ itself

diverges. $\square$

3. Same as 1 and 2, but now for

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+4} .$$

<u>Answer.</u> We have that $\dfrac{n^2}{n^3+4} = \dfrac{1/n}{1+4/n^3} \to 0$

as $n \to \infty$, so the test for divergence is

inconclusive.

By using the limit comparison test with

$b_n = \dfrac{1}{n}$ and $a_n = \dfrac{n^2}{n^3+4}$, we have that

$$\frac{a_n}{b_n} = \frac{n^2}{n^2+4/n} = \frac{1}{1+4/n^3} \to 1 \quad \text{as} \quad n \to \infty,$$

so since $\sum b_n = \infty$, $\sum a_n = \infty$.

Thus $\sum (-1)^{n+1} a_n$ is <u>not</u> absolutely convergent.

To see if $\sum (-1)^{n+1} a_n$ converges, we apply the alternating series test and use the same trick as in problem 1.

We let $f(x) = \dfrac{x^2}{x^3+4}$, so

$$f'(x) = \frac{(x^3+4)(2x) - x^2(3x^2)}{(x^3+4)^2}$$

$$= \frac{8x - x^4}{(x^3+4)^2} = \frac{x(8-x^3)}{(x^3+4)^2} < 0$$

when $x \geq 2$.

Thus $a_{n+1} \leq a_n$ for all $n \geq 2$,

so the series $\sum (-1)^{n+1} a_n$ is conditionally convergent by the alternating series test. $\square$

4. Same as 1, 2, 3 but for $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n^2+4}$

Answer. Since $\frac{n}{n^2+4} \to 0$ as $n \to \infty$,

the test for divergence is inconclusive.

Let $a_n = \frac{n}{n^2+4}$ and $b_n = \frac{1}{n}$.

Then $\frac{a_n}{b_n} = \frac{n}{n+4/n} \to 1$ as $n \to \infty$

and $\sum b_n = \infty$, so $\sum a_n = \infty$ by

the limit comparison test.

Thus $\sum (-1)^{n-1} a_n$ is <u>not</u> absolutely convergent.

Let $f(x) = \frac{x}{x^2+4}$. Then

$$f'(x) = \frac{x^2+4-2x^2}{(x^2+4)^2} = \frac{4-x^2}{(x^2+4)^2} = \frac{(2+x)(2-x)}{(x^2+4)^2} \leq 0$$

when $x \geq 2$. Thus $\frac{n+1}{(n+1)^2+4} \leq \frac{n}{n^2+4}$ for all $n \geq 2$, so $\sum (-1)^n a_n$ converges conditionally by the alternating series test. $\square$

5. Same as 1–4, but for $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{3^n}{2^n n^3}$.

Answer. We use the ratio test.

$$\left| \frac{(-1)^n \frac{3^{n+1}}{2^{n+1}(n+1)^3}}{(-1)^{n-1} \frac{3^n}{2^n n^3}} \right| = \frac{3}{2} \cdot \left(\frac{n}{n+1}\right)^3 \longrightarrow \frac{3}{2} \text{ as}$$

$n \to \infty$. Therefore, by the ratio test, the series diverges.

6. Same as 1–5 but for $\sum_{n=1}^{\infty} \frac{(-9)^n}{n\, 10^{n+1}}$.

Answer Ratio test:

$$\left| \frac{\frac{(-9)^{n+1}}{(n+1)\, 10^{n+2}}}{\frac{(-9)^n}{n\, 10^{n+1}}} \right| = \frac{9}{10} \frac{n}{n+1} \longrightarrow \frac{9}{10} \text{ as } n \to \infty.$$

Therefore, by the ratio test, the series converges absolutely.

7. Same as 1–6, but for $\sum_{n=1}^{\infty} \frac{n\, 5^{2n}}{10^{n+1}}$.

Answer.

$$\left| \frac{\frac{(n+1)\, 5^{2(n+1)}}{10^{n+2}}}{\frac{n\, 5^{2n}}{10^{n+1}}} \right| = \frac{n+1}{n} \cdot \frac{25}{10} \xrightarrow[n\to\infty]{} \frac{25}{10}$$

Therefore, by the ratio test, the series diverges. $\square$