Recitation notes

Jordan Hoffart

August 20, 2024

1 Organizational matters

1. MATH 152 Sections 504/505/506.
2. Canvas page `https://canvas.tamu.edu/courses/331381`.
3. Website `https://jordanhoffart.github.io/teaching/f24m152`
4. Course page `https://www.math.tamu.edu/courses/math152/`
5. Tuesday recitations in HEB 137/222. Thursday labs in BLOC 123/124.
6. Recitations
   (a) Review last week's material.
   (b) Take a quiz.
7. Labs
   (a) Bring your own device.
   (b) Programming in Python.
   (c) More details on Thursday.

2 Fundamental theorem of calculus

Definition 1 (Antiderivative)**.** Let f be a function defined on an interval I. An antiderivative of f is a differentiable function F defined on I such that

$$F'(x) = f(x) \tag{1}$$

for all x in I.

Example 1. Let $f(x) = 3x^2$. Then $F(x) = x^3$ is an antiderivative of f. In fact, for any constant C, $F(x) = x^3 + C$ is an antiderivative of f. One can show that every antiderivative of f is of this form. That is, if F is an antiderivative of f, then there is a constant C such that $F(x) = x^3 + C$.

Theorem 1 (General form of an antiderivative)**.** *If F is an antiderivative of f, then for any constant C, $F + C$ is also an antiderivative of f. Conversely, if F, G are antiderivatives of a function f on an interval I, then there is a constant C such that*

$$F(x) = G(x) + C \tag{2}$$

for all x in I.

Theorem 2 (Fundamental theorem of calculus, part 1)**.** *If f is a continuous function on an interval I containing a point a and we define the function F on I by*

$$F(x) = \int_a^x f(t)\,dt \tag{3}$$

then F is an antiderivative of f on I.

Example 2. Let $f(x) = e^{x^2}$ on $[0, 1]$. Then

$$F(x) = \int_0^x e^{t^2}\,dt \tag{4}$$

is an antiderivative of f, so

$$\frac{d}{dx}\int_0^x e^{t^2}\,dt = \frac{d}{dx}F(x) = f(x) = e^{x^2}. \tag{5}$$

Theorem 3 (Fundamental theorem of calculus, part 2)**.** *If f is a continuous function on an interval $[a, b]$ and if F is an antiderivative of f on $[a, b]$, then*

$$\int_a^b f(x)\,dx = F(b) - F(a). \tag{6}$$

Example 3. Let $f(x) = \cos x$, so that $F(x) = \sin x$ is an antiderivative of f. Then

$$\int_0^\pi \cos x\,dx = \int_0^\pi f(x)\,dx = F(\pi) - F(0) = \sin\pi - \sin 0 = 0. \tag{7}$$

3 Substitution rule

Definition 2 (Indefinite integral)**.** If f is a function with an antiderivative F, then the indefinite integral of f is the collection of all antiderivatives of f. We denote this collection by

$$\int f(x)\,dx. \tag{8}$$

The use of the letter x in our notation is arbitrary. We can use any other symbol, as long as we are consistent. That is, all of these notations represent the indefinite integral of f:

$$\int f(x)\,dx = \int f(y)\,dy = \int f(z)\,dz = \int f(u)\,du = \ldots \tag{9}$$

as well as any other choices of the symbol of integration.

If F is an antiderivative of f, we will abuse notation and denote the indefinite integral of f by

$$\int f(x)\,dx = F(x) + C. \tag{10}$$

Theorem 4 (Substitution rule for indefinite integrals)**.** *If g is a differentiable function on an interval I, and if f is a continuous function on an interval J containing the range $g(I)$ of g, then $G : I \to \mathbb{R}$ is an antiderivative of $(f \circ g)g'$ iff $G = F \circ g$ for some antiderivative $F : J \to \mathbb{R}$ of f. We formally summarize this by writing*

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du \tag{11}$$

with $u = g(x)$. We also summarize this by writing

$$\int f(g(x))g'(x)\,dx = F(g(x)) + C. \tag{12}$$

Proof. Let F be an antiderivate of f and suppose $G = F \circ g$. Then $(F \circ g)'(x) = F'(g(x))g'(x) = f(g(x))g'(x)$, so $G = F \circ g$ is an antiderivative of $(f \circ g)g'$.

Now let G be an antiderivative of $(f \circ g)g'$. We pick a point $a \in J$ and let

$$\widetilde{F}(x) = \int_a^x f(t)\,dt. \tag{13}$$

Then we have that $\widetilde{F}$ is an antiderivative of f. From above, we then have that $\widetilde{F} \circ g$ is an antiderivative of $(f \circ g)g'$. Since G is also an antiderivative of $(f \circ g)g'$, there is a constant C such that

$$G(x) = \widetilde{F}(g(x)) + C \tag{14}$$

for all x in I. Then we let

$$F(x) = \widetilde{F}(x) + C, \tag{15}$$

so that F is an antiderivative of f for which $G = F \circ g$. □

Example 4. Let $f(x) = \cos x$ and $g(x) = x^3$ on $\mathbb{R}$. Then by formally setting $u = g(x)$,

$$\begin{aligned}
\int \cos(x^3)3x^2\,dx &= \int f(g(x))g'(x)\,dx && (16)\\
&= \int f(u)\,du && (17)\\
&= \int \cos u\,du && (18)\\
&= \sin u + C && (19)\\
&= \sin x^3 + C. && (20)
\end{aligned}$$

Theorem 5 (Substitution rule for definite integrals)**.** *If g is a differentiable function on an interval $[a, b]$, and if f is a continuous function on the interval between $g(a)$ and $g(b)$, then*

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du. \tag{21}$$

Example 5. Let $g(x) = \sin x$ on $[0, \pi]$ and let $f(x) = x^2$. Then

$$\begin{aligned}
\int_0^\pi \sin(x)^2 \cos x\,dx &= \int_a^b f(g(x))g'(x)\,dx && (22)\\
&= \int_{g(a)}^{g(b)} f(u)\,du && (23)\\
&= \int_0^0 u^2\,du && (24)\\
&= 0. && (25)
\end{aligned}$$