Quiz 6 Practice Problems Section 11.4

Does the series converge or diverge? Use the comparison test.

14. $\displaystyle\sum_{n\geq 1}^{\infty} \frac{1}{\sqrt[3]{3n^4+1}}$

Answer It converges. To see why, we observe that when n is very large, the series terms "look like"

$$\frac{1}{\sqrt[3]{3n^4}} = \frac{1}{3n^{4/3}}$$

which should give us a convergent series by the p-test.

To make the logic rigorous, we use the comparison test. For $n \geq 1$, we have that

$$3n^4+1 > 3n^4, \quad \text{so}$$

$$\frac{1}{\sqrt[3]{3n^4}} > \frac{1}{\sqrt[3]{3n^4+1}}. \quad \text{Thus,}$$

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{3n^4+1}} \leq \sum_{n=1}^{\infty} \frac{1}{3n^{4/3}} < \infty$$

by the p-test. Thus, the series converges by comparison.

21. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{1+n}}{2+n}$

Answer: It diverges. For large n, the terms "look like" $\frac{\sqrt{n}}{2n} = \frac{1}{2n^{1/2}}$, which diverges by the p-test.

To prove this, we can use either the comparison test or the limit comparison test.

For the comparison test, we have that when $n > 2$, $2+n < 2n$, so $\frac{1}{2n} < \frac{1}{2+n}$. We also have that $\sqrt{n+1} > \sqrt{n}$.

Therefore, $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{1+n}}{2+n} =$

$$\underbrace{\sum_{n=1}^{2} \frac{\sqrt{1+n}}{2+n}}_{\geq 0} + \sum_{n=3}^{\infty} \frac{\sqrt{1+n}}{2+n}$$

$$\geq \sum_{n=3}^{\infty} \frac{\sqrt{n}}{2n} = \frac{1}{2} \sum_{n=3}^{\infty} \frac{1}{n^{1/2}} = \infty$$

by the p-test. Thus, the series diverges by comparison.

If, instead, we wanted to use the limit comparison test, then we let $a_n = \dfrac{\sqrt{1+n}}{2+n}$ and we ~~would~~ need to find a sequence $b_n > 0$ such that

$$\frac{a_n}{b_n} = \frac{\sqrt{1+n}}{b_n(2+n)} \to \infty \quad \text{as } n \to \infty$$

and $\displaystyle\sum_{n=1}^{\infty} b_n = \infty$.

If we set $b_n = \frac{1}{n}$, then $\displaystyle\sum_{n=1}^{\infty} b_n = \infty$ by the p-test and

$$\frac{a_n}{b_n} = \frac{\sqrt{1+n}}{2/n + 1} \to \infty \quad \text{as } n \to \infty,$$

so we conclude that $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{1+n}}{2+n} = \infty$ by the limit comparison test. $\square$