Section 7.4 Notes

Theorem 1

Every polynomial is a product of linear terms $ax+b$ and irreducible quadratic terms cx^2+dx+e with $d^2-4ce<0$.

Theorem 2 If $p(x), q(x)$ are 2 polynomials with $\deg p \geq \deg q$, then we can find polynomials $s(x), r(x)$ with

$$\frac{p(x)}{q(x)} = s(x) + \frac{r(x)}{q(x)}$$

where $\deg s = \deg p - \deg q$ and $\deg r < \deg q$.

Partial Fraction Decomposition

Given two polynomials $p(x), q(x)$

1. If $\deg p \geq \deg q$, use polynomial long division to write

$$\frac{p(x)}{q(x)} = s(x) + \frac{r(x)}{q(x)}$$

as in Theorem 2.

2. If $\deg p < \deg q$, take $s(x) = 0$, $r(x) = p(x)$.

3. Use various factoring techniques to write

$$q(x) = (a_1x+b_1)^{d_1}(a_2x+b_2)^{d_2}\cdots(a_nx+b_n)^{d_n}\cdot$$
$$(c_1x^2+e_1x+f_1)^{\tilde{d}_1}(c_2x^2+e_2x+f_2)^{\tilde{d}_2}\cdots(c_mx^2+e_mx+f)^{\tilde{d}_m}$$

as a product of linear terms a_ix+b_i and irreducible quadratic terms

$$c_ix^2+e_ix+f_i \quad \text{with} \quad e_i^2-4c_if_i<0$$

as in Theorem 1.

4. Depending on which terms show up in the factorization of q

(i) $q(x) = (ax+b)^n \longrightarrow$

$$\frac{r(x)}{q(x)} = \frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \frac{A_3}{(ax+b)^3} + \cdots + \frac{A_n}{(ax+b)^n}$$

for some $A_i \in \mathbb{R}$.

(ii) $q(x) = (ax^2+bx+c)^n$ with $b^2-4ac<0$

then
$$\frac{r(x)}{q(x)} = \frac{A_1x+B_1}{ax^2+bx+c} + \cdots + \frac{A_nx+B_n}{(ax^2+bx+c)^n}$$

for some $A_i, B_i \in \mathbb{R}$.

(iii) $q(x) = (a_1x+b_1)^{n_1}(a_2x+b_2)^{n_2} \rightarrow$

$$\frac{r(x)}{q(x)} = \frac{A_1}{a_1x+b_1} + \cdots + \frac{A_{n_1}}{(a_1x+b_1)^{n_1}} +$$

$$\frac{B_1}{a_2x+b_2} + \cdots + \frac{B_{n_2}}{(a_2x+b_2)^{n_2}}$$

ie do (i) to both pieces $(a_ix+b_i)^{n_i}$ and add

(iv) $q(x) = (a_1x^2+b_1x+c_1)^{n_1}(a_2x^2+b_2x+c_2)^{n_2}$

with $b_i^2-4a_ic_i<0$

Then, like in (iii), do (ii) to both pieces

(v) $q(x) = (ax+b)^n(cx^2+dx+e)^m$ with $d^2-4ce<0$

Then do (i) to and (ii) to and add

Any other case follows a similar pattern.

7.3 Solutions

1a $$\frac{4+x}{(1+2x)(3-x)} = \frac{A}{1+2x} + \frac{B}{3-x} \quad \text{for some } A, B \in \mathbb{R}.$$

1b $$\frac{1-x}{x^3+x^4} = \frac{1-x}{x^3(1+x)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{1+x}$$

for some $A, B, C, D \in \mathbb{R}$.

2a $$\frac{x-6}{x^2+x-6} = \frac{x-6}{(x+3)(x-2)} = \frac{A}{x+3} + \frac{B}{x-2}$$

for some $A, B \in \mathbb{R}$.

2b $$\frac{x^2}{\underbrace{x^2+x+6}} = 1 - \frac{x+6}{x^2+x+6}$$

irreducible : discriminant $1^2 - 4 \cdot 1 \cdot 6 < 0$

polynomial long division

$$\begin{array}{r l} & \quad 1 \\ x^2+x+6 & \overline{)\,1x^2} \\ & -(x^2+x+6) \\ \hline & \quad -x-6 \end{array}$$

3a $$\frac{1}{x^2+x^4} = \frac{1}{x^2(1+x^2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{1+x^2}$$

for some $A, B, C, D \in \mathbb{R}$.

3b $$\frac{x^3+1}{x^3-3x^2+2x} = 1 + \frac{3x^2-2x+1}{x^3-3x^2+2x} = 1 + \frac{3x^2-2x+1}{x(x^2-3x+2)}$$

$$= 1 + \frac{3x^2-2x+1}{x(x-2)(x-1)}$$

polynomial long division

$$\begin{array}{r|l} & \quad 1 \\ x^3-3x^2+2x & x^3+1 \\ & -(x^3-3x^2+2x) \\ \hline & \quad 3x^2-2x+1 \end{array}$$

$$= 1 + \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x-1}$$ for some $A, B, C \in \mathbb{R}$.

4a $$\frac{x^4-2x^3+x^2+2x-1}{x^2-2x+1} = x^2 + \frac{2x-1}{x^2-2x+1}$$

$$\begin{array}{r|l} & x^2 \\ \hline x^2-2x+1 & x^4-2x^3+x^2+2x-1 \\ & -(x^4-2x^3+x^2) \\ \hline & 2x-1 \end{array}$$

$$= x^2 + \frac{2x-1}{(x-1)^2} = x^2 + \frac{A}{x-1} + \frac{B}{(x-1)^2}.$$

4b $$\frac{x^2-1}{x^3+x^2+x} = \frac{(x+1)(x-1)}{x\underbrace{(x^2+x+1)}_{\text{irreducible}}} = \frac{A}{x} + \frac{Bx+C}{x^2+x+1}$$

5a

$$\frac{x^6}{x^2-4} = x^4 + 4x^2 + 16 + \frac{64}{(x+2)(x-2)}$$

$$\begin{array}{r} x^4 + 4x^2 + 16 \\ x^2-4 \,\big)\, x^6 \qquad\qquad \\ -(x^6 - 4x^4) \qquad \\ \hline 4x^4 \qquad \\ -(4x^4 - 16x^2) \\ \hline 16x^2 \\ -(16x^2 - 64) \\ \hline 64 \end{array}$$

$$= x^4 + 4x^2 + 16 + \frac{A}{x+2} + \frac{B}{x-2}$$

5b

$$\frac{x^4}{\underbrace{(x^2-x+1)}_{\text{irreducible}}(x^2+2)^2} = \frac{Ax+B}{x^2-x+1} + \frac{Cx+D}{x^2+2} + \frac{Ex+F}{(x^2+2)^2}$$

$$\begin{aligned} t^3+1 &= (at+b)(ct^2+dt+e) \\ &= act^3 + (ad+bc)t^2 \\ &\quad + (ae+bd)t + be \end{aligned}$$

$ac=1,\ be=1 \qquad a=c=b=e=1$

$ad+bc=0 \rightarrow \quad d=-1$

$ae+bd=0$

$t^3+1 = (t+1)(t^2-t+1)$

$\boxed{6a}$

Difficult!

$$\begin{array}{r} 1 \qquad \\ t^6+t^3 \,\big)\, t^6+1 \quad \\ -(t^6+t^3) \\ \hline -t^3+1 \end{array}$$

$$\frac{t^6+1}{t^6+t^3} = 1 + \frac{1-t^3}{t^6+t^3} = 1 + \frac{1-t^3}{t^3(t^3+1)}$$

$$= 1 + \frac{A}{t} + \frac{B}{t^2} + \frac{C}{t^3} + \frac{D}{t+1} + \frac{Et+F}{t^2-t+1}$$

$\boxed{6b}$

Class

$$\frac{x^5+1}{(x^2-x)(x^4+2x^2+1)} = \frac{x^5+1}{x(x-1)(x^2+1)^2}$$

$$= \frac{A}{x} + \frac{B}{x-1} + \frac{Cx+D}{x^2+1} + \frac{Ex+F}{(x^2+1)^2}$$

7.

$$\begin{array}{r|l} & x^3 + x^2 + x + 1 \\ x-1 & x^4 \\ & -(x^4 - x^3) \\ \hline & x^3 \\ & -(x^3 - x^2) \\ \hline & x^2 \\ & -(x^2 - x) \\ \hline & x \\ & -(x-1) \\ \hline & 1 \end{array}$$

$$\int \frac{x^4}{x-1}\,dx = \int x^3 + x^2 + x + 1 + \frac{1}{x-1}\,dx$$

$$= \frac{1}{4}x^4 + \frac{1}{3}x^3 + \frac{1}{2}x^2 + x + \ln|x-1| + C$$

8 $\displaystyle\int \frac{3t-2}{t+1}\,dt = \int 3 - \frac{5}{t+1}\,dt = 3t - 5\ln|t+1| + C$

$$\begin{array}{r} 3 \\ t+1 \overline{\big)\, 3t-2} \\ -(3t+3) \\ \hline -5 \end{array}$$

9 $\displaystyle\frac{5x+1}{(2x+1)(x-1)} = \frac{A}{2x+1} + \frac{B}{x-1}$ for some $A, B \rightarrow$

$5x+1 = A(x-1) + B(2x+1)$ for all x

$x=1 \longrightarrow 6 = 3B \rightarrow B = 2$

$2x+1=0 \rightarrow x=-1/2 \rightarrow 1-5/2 = A \rightarrow A = -1/2$

so $\displaystyle\int \frac{5x+1}{(2x+1)(x-1)}\,dx = -\frac{1}{2}\int \frac{1}{2x+1}\,dx + 2\int \frac{1}{x-1}\,dx$

$\displaystyle = -\frac{1}{2}\ln|2x+1| + 2\ln|x-1| + C$

10 $\displaystyle\int \frac{y}{(y+4)(2y-1)}\,dy = \int \frac{A}{y+4} + \frac{B}{2y-1}\,dy$

$= A\ln|y+4| + B\ln|2y-1| + C$

where $y = A(2y-1) + B(y+4)$ for all y

$\rightarrow$ $2A + B = 1 \rightarrow 9B = 1 \rightarrow B = 1/9$

$4B - A = 0 \rightarrow A = 4B \rightarrow A = 4/9$

$= \frac{4}{9}\ln|y+4| + \frac{1}{9}\ln|2y-1| + C$

21 $\displaystyle\int \frac{dt}{(t^2-1)^2} = \int \frac{dt}{(t+1)^2(t-1)^2} = \int \frac{A}{t+1} + \frac{B}{(t+1)^2} + \frac{C}{t-1} + \frac{D}{(t-1)^2}\,dt$

$= A\ln|t+1| - \frac{B}{t+1} + C\ln|t-1| - \frac{D}{t-1} + \text{const.}$

where $1 = A(t+1)(t-1)^2 + B(t-1)^2 + C(t-1)(t+1)^2 + D(t+1)^2$

for all t $t=1 \rightarrow 4D = 1 \rightarrow \boxed{D = 1/4}$

$t=-1 \rightarrow \boxed{B = 1/4}$

$t=0 \rightarrow \quad 1 = A + 1/4 - C + \frac{1}{4}$

$\rightarrow \boxed{A - C = 1/2}$

$t = 2 \rightarrow \quad 1 = 3A + 1/4 + 9C + 9/4$

$\rightarrow \quad -6/4 = 3(A+C) \rightarrow \boxed{-\frac{1}{2} = A + C}$

$\boxed{A = 0 \, , \, C = -1/2}$

So $\displaystyle\int \frac{1}{(t^2-1)^2}\,dt = -\frac{1}{4(t+1)} - \frac{1}{2}\ln|t+1| - \frac{1}{4(t-1)} + \text{const.}$

$\boxed{24}$ Class

$$\int \frac{x^2-x+6}{x^3+3x}\,dx = \int \frac{x^2-x+6}{x(x^2+3)}$$

$$= \int \frac{A}{x} + \frac{Bx+C}{x^2+3}\,dx = A\ln|x| + B\int \frac{x}{x^2+3}\,dx$$

$$+ \frac{C}{3}\int \frac{1}{\left(\frac{x}{\sqrt{3}}\right)^2+1}\,dx$$

$u = x^2+3 \rightarrow du = 2x\,dx$

$$= A\ln|x| + \frac{B}{2}\int \frac{1}{u}\,du + \frac{C}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)$$

$$= A\ln|x| + \frac{B}{2}\ln|x^2+3| + \frac{C}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + \text{const}$$

where $A(x^2+3) + Bx^2 + Cx = x^2 - x + 6$ for all x

$\rightarrow$ $A+B = 1 \rightarrow B = -1$

$3A = 6 \rightarrow A = 2$

$C = -1$

Thus :

$$\int \frac{x^2 - x + 6}{x^3 + 3x}\, dx =$$

$$2 \ln|x| - \frac{1}{2} \ln|x^2+3| - \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + \text{const}$$