Getting Started with Proofs

A <u>proof</u> is a rigorous, logically sound, mathematical argument. It is the means by which we convince ourselves and others that a mathematical statement is true.

<u>Example</u> Suppose we want to prove the following statement:

"If n is an even integer, then n^2 is also an even integer."

Before we even attempt to show this is true, we need to decide what our <u>axioms</u> (mathematical facts that we just agree are true without proof) and what our <u>definitions</u> are. For this particular setting, our axioms are the following:

1. There is a collection of numbers called "integers". They include numbers such as $0, 1, 2, \ldots$ and $-1, -2, -3, \ldots$ and nothing else.

2. The usual arithmetic operations (rules involving adding, subtracting, multiplying, and dividing) integers are true.

For our definitions, we need to know what it means precisely for an integer to be "even". We want this definition to include all of our familiar examples of even integers such as $0, 2, 4, 6, 8, 10, 12, \ldots$
$-2, -4, -6, -8, -10, -12, \ldots$

All of our examples can be written in the form $2 \cdot k$ where k is another integer, eg $2 = 2 \cdot 1$, $10 = 2 \cdot 5$, $-12 = 2 \cdot (-6)$.

Thus, we should make the following definition:

Definition An integer n is even if there is another integer k such that $n = 2k$.

Now that we have our axioms and definitions, let us return to and prove the statement we wanted to prove. Another word for a mathematical statement is proposition, theorem, or lemma.

<u>Proposition</u> : "If n is an even integer, then n^2 is also an even integer."

proof 1 (version for teaching purposes).

We always start with our assumptions.

Step 1. Let n be an even integer.

Now we string together any relevant definitions and axioms in a logical manner until we reach the desired conclusion : "n^2 is an even integer".

Step 2. Then $n = 2k$ for some integer k

(definition of "even").

Step 3. Thus $n^2 = (2k)^2 = 4k^2 = 2(2k^2)$.

$$n^2 \underset{\substack{\uparrow \\ \text{(step 2)}}}{=} (2k)^2 \underset{\uparrow}{=} 4k^2 \underset{\uparrow}{=} 2(2k^2)$$

(step2)

axioms of arithmetic for integers

Step 4. Since k is an integer, $2k^2$ is an integer (follows from axioms of arithmetic for integers)

Step 5. Thus, from step 3 and 4, we have

then $$n^2 \underset{\text{Step 3}}{=} 2 \cdot \underbrace{2k^2}_{\text{is an integer by step 4}}$$

Conclusion: By the definition of being an even integer, step 5 shows that n^2 is an even integer. $\square$

symbol to denote the end of the proof.

We have successfully written a proof. This example illustrates the basic layout of a mathematical proof. We describe it more generally below.

Structure of a proof:

Proposition: Some assumptions imply some conclusions.

denotes the statement to be proved

proof.

denotes the start of a proof

Assumptions } beginning

Logical chain of axioms | definitions | other theorems connecting the assumptions to the conclusions. } middle

Conclusions. } end

$\square$

denotes end of proof

As an exercise, go back through the proof 1 in the example and identify which parts of the proof fit into the general structure given above.

The way that proof 1 was written was to clearly illustrate the basic layout of a proof. In practice, most proofs do not spell out as many details, nor do they label steps (unless the proof is really long and complicated). there is how a more experienced mathematician would rewrite proof 1.

Proposition: If n is an even integer, then n^2 is an even integer.

proof. Suppose n is an even integer. Then $n = 2k$ for some integer k. Therefore, $n^2 = (2k)^2 = 4k^2 = 2 \cdot (2k^2)$. Since $2k^2$ is an integer, we conclude that n^2 is even. $\square$

Notice that this proof is shorter, does not state when it is invoking an axiom or definition, is written in complete sentences in paragraph format, and does not label its steps.

There is no universal or definitive set of rules on what makes a proof good or bad. The following list includes some things that most would agree that a good proof should possess:

1. Every step <u>must</u> be true. This is non-negotiable. Using one false step in a proof no longer makes it valid.

2. Assumptions and conclusions should be clearly stated.

3. Complete, grammatically correct sentences <u>must</u> be used. This is also a non-negotiable.

4. **All** logic must be sound. It is not enough to write a bunch of true sentences. These should be logically related to one another and should connect the assumptions to the conclusions.

Example:

Good - Suppose it is raining outside. Then it is wet.

Bad - Suppose all cats are mammals. Then the sun is a star. Therefore, blue is a color. In conclusion, the earth is a planet.

5. There should be no leaps in logic. Even if it is possible to show one statement follows from another, if too many intermediate steps are left out, then the argument is hard to follow and not very convincing. In other words, don't write too little.

6. At the same time, one should not spell out

every little detail, nor mention every time an axiom or definition is used, nor give too wordy of explanations. In other words, don't write too much.

There are many more, but this should give a reasonable guideline to follow when writing proofs. Like any skill, proof-writing is best learned by practicing. One should just start writing proofs, and then get them critiqued by someone more experienced. This will lead to the most improvement over time.