HW 3 Hints

1. For a smooth function $u : \mathbb{R}^n \to \mathbb{R}$, its kth order Taylor expansion near $x \in \mathbb{R}^n$ is given by

$$u(x+h) = \sum_{\substack{\alpha \in \mathbb{N}^n \\ |\alpha| \leq k}} \frac{h^\alpha}{\alpha!} D^\alpha u(x) + O(|h|_\infty^{k+1})$$

for any $h = (h_1, \dots, h_n) \in \mathbb{R}^n$. The sum is over all multi-indices $\alpha = (\alpha_1, \dots, \alpha_n)$ with each $\alpha_i \in \mathbb{N} := \{0, 1, 2, \dots\}$ such that the order of the index $|\alpha| := \alpha_1 + \dots + \alpha_n \leq k$.

For a vector $h = (h_1, \dots, h_n) \in \mathbb{R}^n$ and a multiindex $\alpha \in \mathbb{N}^n$, $h^\alpha := h_1^{\alpha_1} h_2^{\alpha_2} \cdots h_n^{\alpha_n} \in \mathbb{R}$.

For a smooth function $u : \mathbb{R}^n \to \mathbb{R}$, a multi-index $\alpha \in \mathbb{N}^n$, and a point $x \in \mathbb{R}^n$, the α mixed partial derivative of u at x is

$$D^\alpha u(x) := \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} u(x).$$

For a vector $h \in \mathbb{R}^n$, $|h|_\infty = \max\{|h_1|, \dots, |h_n|\}$.

For a multi-index $\alpha \in \mathbb{N}^n$, $\alpha! := \alpha_1! \alpha_2! \cdots \alpha_n!$

As an example, for $n=1$, $k=3$, we have

$$u(x+h) = u(x) + h\,u'(x) + \frac{h^2}{2}u''(x) + \frac{h^3}{3!}u'''(x) + O(h^4)$$

for any $h \in \mathbb{R}^n$.

For $n=2$, $k=5$, $x = (x_1, x_2)$, $h = (h_1, h_2)$,

we have

$$u(x_1+h_1, x_2+h_2) = u(x_1,x_2) + h_1\frac{\partial}{\partial x_1}u(x_1,x_2) + h_2\frac{\partial}{\partial x_2}u(x_1,x_2) +$$

$$\frac{h_1^2}{2}\frac{\partial^2}{\partial x_1^2}u(x_1,x_2) + h_1h_2\frac{\partial^2}{\partial x_1\partial x_2}u(x_1,x_2) +$$

$$\frac{h_2^2}{2}\frac{\partial^2}{\partial x_2^2}u(x_1,x_2) +$$

$$\frac{h_1^3}{3!}\frac{\partial^3}{\partial x_1^3}u(x_1,x_2) + \frac{h_1^2h_2}{2}\frac{\partial^3}{\partial x_1^2\partial x_2}u(x_1,x_2) + \frac{h_1h_2^2}{2}\frac{\partial^3}{\partial x_1\partial x_2^2}u(x_1,x_2)$$

$$+ \frac{h_1^3}{3!}\frac{\partial^3}{\partial x_2^3}u(x_1,x_2) + \frac{h_1^4}{4!}\frac{\partial^4}{\partial x_1^4}u(x_1,x_2) +$$

$$\frac{h_1^3h_2}{3!}\frac{\partial^4}{\partial x_1^3\partial x_2}u(x_1,x_2) + \frac{h_1^2h_2^2}{2\cdot 2}\frac{\partial^4}{\partial x_1^2\partial x_2^2}u(x_1,x_2) +$$

$$\frac{h_1h_2^3}{3!}\frac{\partial^4}{\partial x_1\partial x_2^3}u(x_1,x_2) + \frac{h_2^4}{4!}\frac{\partial^4}{\partial x_2^4}u(x_1,x_2) +$$

$$\frac{h_1^5}{5!}\frac{\partial^5}{\partial x_1^5}u(x_1,x_2) + \frac{h_1^4 h_2}{4!}\frac{\partial^5}{\partial x_1^4 \partial x_2}u(x_1,x_2) +$$

$$\frac{h_1^3 h_2^2}{3!\cdot 2}\frac{\partial^5}{\partial x_1^3 \partial x_2^2}u(x_1,x_2) + \frac{h_1^2 h_2^3}{2\cdot 3!}\frac{\partial^5}{\partial x_1^2 \partial x_2^3}u(x_1,x_2) +$$

$$\frac{h_1 h_2^4}{4!}\frac{\partial^5}{\partial x_1 \partial x_2^4}u(x_1,x_2) + \frac{h_2^5}{5!}\frac{\partial^5}{\partial x_2^5}u(x_1,x_2) + O(\max\{h_1^6, h_2^6\}).$$

2. Your answer should of the form:

Find $u \in V$ such that

$$c_1 u'' + c_2 u' + c_3 u = f \quad \text{on } (0,1)$$
$$\alpha_1 u(0) + \alpha_2 u'(0) = 0 \qquad , \quad \text{where}$$
$$\beta_1 u(1) + \beta_2 u'(1) = 0$$

V is some space of functions on $[0,1]$, and

$c_1, c_2, c_3, \alpha_1, \alpha_2, \beta_1, \beta_2$ are constants. It is your job to find what $V, c_1, c_2, c_3, \alpha_1, \alpha_2, \beta_1, \beta_2$ should be.

3. Your answer should be of the form:

Find $u \in V$ such that

$$A(u,v) = \int_0^1 fv \quad \text{for all}$$

$v \in V$, where V is a space of functions on $[0,1]$ and $A(u,v)$ is some expression involving integrals of terms involving u, v, u', v', and γ.

It is your job to find A and V.

4. Your answer should be of the form:

Find $u \in V$ st $B(u,v) = \int_\Omega fv + \int_\Gamma gv$

for all $v \in V$, where

$B(u,v)$ is some expression involving integrals of terms involving A, ∇u, ∇v, u, v, and γ.

Some Notation / Results from vector Calculus:

Given $u: \mathbb{R}^n \to \mathbb{R}$, the gradient of u

$\nabla u : \mathbb{R}^n \to \mathbb{R}^n$ is defined by

$$\nabla u(x) = \left(\frac{\partial}{\partial x_1} u(x), \dots, \frac{\partial}{\partial x_n} u(x) \right)$$

Given $V : \mathbb{R}^n \to \mathbb{R}^n$ whose component functions are $V = (V_1, \dots, V_n)$, the divergence of V $\nabla \cdot V : \mathbb{R}^n \to \mathbb{R}$ is given by $\nabla \cdot V(x) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} V_i(x)$.

Given $u : \mathbb{R}^n \to \mathbb{R}^n$ and $v : \mathbb{R}^n \to \mathbb{R}^n$, their dot product is $u \cdot v : \mathbb{R}^n \to \mathbb{R}$ given by $(u \cdot v)(x) = u(x) \cdot v(x) = \sum_{i=1}^{n} u_i(x) v_i(x)$

Product Rule

If $\varphi : \mathbb{R}^n \to \mathbb{R}$ and $v : \mathbb{R}^n \to \mathbb{R}^n$, then

$$\nabla \cdot (\varphi v) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} (\varphi v_i) = \sum_{i=1}^{n} \left(\frac{\partial}{\partial x_i} \varphi \right) v_i + \sum_{i=1}^{n} \varphi \frac{\partial}{\partial x_i} v_i$$

$$\rightarrow \quad \boxed{\nabla \cdot (\varphi v) = (\nabla \varphi) \cdot v + \varphi (\nabla \cdot v)}$$

In coordinates:

$$\boxed{\sum_{i=1}^{n} \frac{\partial}{\partial x_i} (\varphi v_i) = \sum_{i=1}^{n} \left(\frac{\partial}{\partial x_i} \varphi \right) v_i + \varphi \sum_{i=1}^{n} \frac{\partial}{\partial x_i} v_i}$$

The Divergence Theorem

Given $v : \mathbb{R}^n \to \mathbb{R}^n$ and $\Omega \subset \mathbb{R}^n$ with boundary Γ and outward unit normal $n : \Gamma \to \mathbb{R}^n$

$$\boxed{\int_\Omega \nabla \cdot v = \int_\Gamma v \cdot n}$$

In coordinates:

$$\int_\Omega \sum_{i=1}^{n} \frac{\partial}{\partial x_i} v_i = \int_\Gamma \sum_{i=1}^{n} v_i n_i$$

The Divergence Theorem + Product Rule give us

integration by parts in $\mathbb{R}^n$:

Given $\varphi : \mathbb{R}^n \to \mathbb{R}$, $v : \mathbb{R}^n \to \mathbb{R}^n$,

$\Omega \subset \mathbb{R}^n$ w/ boundary Γ and outward unit normal n,

$$\int_\Omega \nabla \cdot (\varphi v) \underset{\substack{\uparrow \\ \text{product}}}{=} \int_\Omega \nabla \varphi \cdot v + \int_\Omega \varphi (\nabla \cdot v) \quad \text{and}$$

$$\int_\Omega \nabla \cdot (\varphi v) \underset{\substack{\uparrow \\ \text{divergence}}}{=} \int_\Gamma \varphi v \cdot n \quad , \quad \text{so}$$

$$\boxed{\int_\Omega \varphi \nabla \cdot v = \int_\Gamma \varphi v \cdot n - \int_\Omega \nabla \varphi \cdot v}$$

Integration by parts in $\mathbb{R}^n$ w/ scalar function φ and vector function v.

You can rewrite the PDE as

$$\begin{cases} -\nabla \cdot (A \nabla u) + \gamma u = f & \text{on } \Omega \\ (A \nabla u) \cdot n = g & \text{on } \Gamma . \end{cases}$$