Finite Differences

Consider the following ODE:

$$\begin{cases} -p\,u''(x) + q\,u'(x) + r\,u(x) = f(x), & a < x < b \\ u(a) = g_a \\ u(b) = g_b \end{cases}$$

We wish to numerically approximate the solution u to this problem. The key mathematical tool we are going to use is Taylor's Theorem:

If $u : (a,b) \to \mathbb{R}$ has $k+1$ continuous derivatives at $x \in (a,b)$, then for all sufficiently small h,

$$u(x+h) = \underbrace{u(x) + u'(x)h + u''(x)\frac{h^2}{2} + \cdots + u^{(k)}(x)\frac{h^k}{k!}}_{P_h(x)} + O(h^{k+1})$$

where the $O(h^{k+1})$ simply means that there is a constant C so

$$|u(x+h) - P_h(x)| \le C h^{k+1}$$ when h is sufficiently small.

How do we use this? We want to replace the derivatives in our PDE by certain differences involving $u(x+h)$ and $u(x)$ for small h. From Taylor's theorem with $k=1$, we have

$$u(x+h) = u(x) + u'(x)h + O(h^2) \longrightarrow$$

$$\frac{u(x+h) - u(x)}{h} = u'(x) + O(h) .$$

This can be interpreted as saying that we can replace $u'(x)$ by $\frac{u(x+h)-u(x)}{h}$ and we will introduce an error that scales like h. This is known as a first order approximation of $u'(x)$.

Now let's approximate $u''(x)$. We will once again use Taylor's Theorem, but in a more clever way. From Taylor's theorem with $k=3$

$$u(x+h) = u(x) + u'(x)h + u''(x)\frac{h^2}{2} + u'''(x)\frac{h^3}{6} + O(h^4)$$

We can also replace h by $-h$ in the above to get

$$u(x-h) = u(x) - u'(x)h + u''(x)\frac{h^2}{2} - u'''(x)\frac{h^3}{6} + O(h^4)$$

Adding these two equations gives us

$$u(x+h) + u(x-h) = 2u(x) + u''(x)h^2 + O(h^4) \rightarrow$$

$$\frac{u(x+h) - 2u(x) + u(x-h)}{h^2} = u''(x) + O(h^2).$$

This says that we can replace $u''(x)$ by

$\dfrac{u(x+h) - 2u(x) + u(x-h)}{h^2}$ and we will only

introduce an error of order h^2. This is known as a <u>second order</u> approximation to $u''(x)$.

To summarize :

$$u'(x) = \frac{u(x+h) - u(x)}{h} + O(h)$$

$$u''(x) = \frac{u(x+h) - 2u(x) + u(x-h)}{h^2} + O(h^2)$$

Let us substitute these into our ODE:

$$\left[-p\left(\frac{u(x+h)-2u(x)+u(x-h)}{h^2} + O(h^2)\right) + q\left(\frac{u(x+h)-u(x)}{h} + O(h)\right) + r\,u(x) \right] = f(x)$$

The left side can be rewritten as

$$\underbrace{-\frac{p}{h^2}}_{:=\alpha} u(x-h) + \underbrace{\left[\frac{2p}{h^2} - \frac{q}{h} + r\right]}_{:=\beta} u(x) + \underbrace{\left[\frac{q}{h} - \frac{p}{h^2}\right]}_{:=\gamma} u(x+h) + \underbrace{\left(-p\,O(h^2) + q\,O(h)\right)}_{=O(h)}$$

so that

$$\alpha\, u(x-h) + \beta u(x) + \gamma\, u(x+h) + O(h) = f(x) .$$

This says that, for sufficiently small h, our solution u to our ODE satisfies the equation

$$\alpha u(x-h) + \beta u(x) + \gamma u(x+h) = f(x)$$

up to an approximation error that is of size h.

This is known as the

<u>1st order forward finite difference approximation</u>

to our ODE at the point x.

So far we have shown that if u solves our ODE, then u approximately satisfies the finite difference equation at each $a < x < b$. We are now going to construct a function u_h that solves the finite difference equation at finitely many points $a < x_1 < x_2 < \dots < x_N < b$ and which also satisfies the boundary conditions $u_h(a) = g_a$, $u_h(b) = g_b$. This u_h will be a good approximation of the original solution u.

To start, let $k > 0$ and let $h = \dfrac{b-a}{2^k}$.

Now let $x_i = a + ih$ for $i = 0, 1, \dots, 2^k$.

Let $u_h : [a,b] \to \mathbb{R}$ be piecewise linear on each

$[x_i, x_{i+1}]$ and satisfy the finite difference equation

$$\alpha\, u(x_i - h) + \beta\, u(x_i) + \gamma\, u(x_i + h) = f(x_i)$$

for each $i = 1, 2, \ldots, 2^k - 1$ as well as the boundary conditions

$$u(a) = u(x_0) = g_a$$

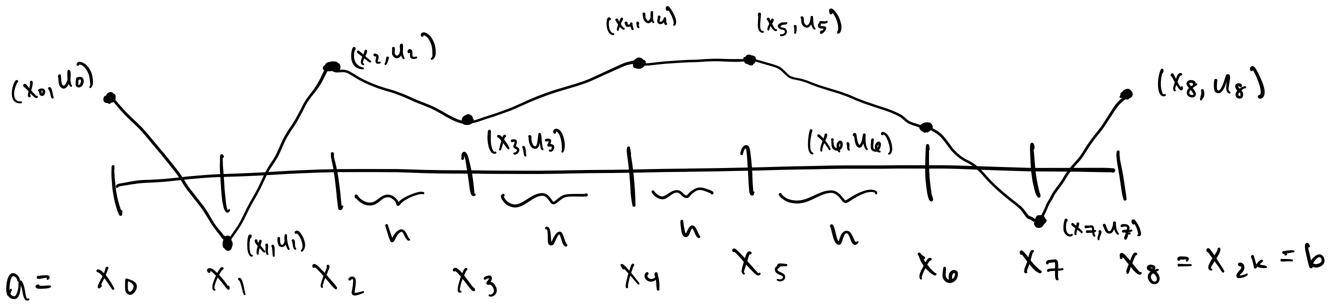
$$u(b) = u(x_{2^k}) = g_b$$

For notation, let $u_i = u(x_i)$ and $f_i = f(x_i)$.

Since $x_i - h = x_{i-1}$ and $x_i + h = x_{i+1}$, we have the following system of equations:

$$
\begin{aligned}
u_0 \quad & = g_a \\
\alpha u_0 + \beta u_1 + \gamma u_2 \quad & = f_1 \\
\alpha u_1 + \beta u_2 + \gamma u_3 \quad & = f_2 \\
\ddots \quad & \\
\alpha u_{2^k-2} + \beta u_{2^k-1} + \gamma u_{2^k} & = f_{2^k-1} \\
u_{2^k} & = g_b
\end{aligned}
$$

There are 2^k+1 linear equations in 2^k+1 unknowns $u_0, u_1, \ldots, u_{2^k}$. We can solve this system for the $u_0, u_1, \ldots, u_{2^k}$. Since u_h is piecewise linear on each $[x_i, x_{i+1}]$, these 2^k+1 values uniqulty determine u_h. For example, u_h looks like (here $k=3$)



To Summarize the Finite Difference Method:

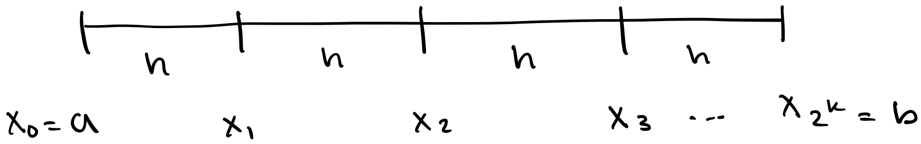
1. Discretize the PDE with finite differences

$$\begin{cases} -pu''(x) + qu'(x) + ru(x) = f(x) \\ u(a) = g_a \\ u(b) = g_b \end{cases} \longrightarrow$$

$$\begin{cases} -p\left(\dfrac{u(x+h) - 2u(x) + u(x-h)}{h^2}\right) + q\left(\dfrac{u(x+h) - u(x)}{h}\right) + ru(x) = f(x) \\ u(a) = g_a \\ u(b) = g_b \end{cases}$$

2. Discretize the domain

$h = \frac{b-a}{2^k}$



$x_i = a + ih \qquad i = 0, 1, \dots, 2^k$

3. Set up linear system

$$\begin{cases} u_0 = g_a \\ -p\left(\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2}\right) + q\left(\frac{u_{i+1} - u_i}{h}\right) + ru_i = f_i \qquad i = 1, \dots, 2^k - 1 \\ u_{2^k} = g_b \end{cases}$$

$u_i = u(x_i)$

$f_i = f(x_i)$

4. Solve linear system for $u_0, u_1, \dots, u_{2^k}$

<u>Remark</u> In step 1, we used a 2nd order approximation for u'' but only a 1st order approximation for u'. This reduces the accuracy

of our method to only 1^{st} order. Is there a way to approximate u'' to the second order, thus boosting the accuracy of our method by 1 degree higher?. Yes! Here's how.

Start with Taylor's Theorem with $k=2$, h, and $-h$:

$$(1) \quad u(x+h) = u(x) + u'(x)h + u''(x)\frac{h^2}{2} + O(h^3)$$

$$(2) \quad u(x-h) = u(x) - u'(x)h + u''(x)\frac{h^2}{2} + O(h^3)$$

Subtract equation 2 from equation 1:

$$u(x+h) - u(x-h) = 2u'(x)h + O(h^3) \longrightarrow$$

$$\frac{u(x+h) - u(x-h)}{2h} = u'(x) + O(h^2).$$

This gives us a <u>2nd order centered</u>

approximation of $u'(x)$. Replace $u'(x)$

with $\frac{u(x+h) - u(x-h)}{2h}$ in step 1 to get

a 2nd order method. This will modify what

your linear system looks like in step 3.

I leave the details to you as an exercise.

Some final remarks:

1. We can also discretize $u'(x)$ via

$$u'(x) = \frac{u(x) - u(x-h)}{h} + O(h)$$

This known as a <u>1st order backwards difference</u>.

2. In our original ODE

$$-pu''(x) + qu'(x) + ru(x) = f(x)$$

the middle term $qu'(x)$, known as the

advection term, gives rise to some common terminology:

(i) If $q > 0$:

- The forward approximation

$$q\,u'(x) \approx q\left(\frac{u(x+h) - u(x)}{h}\right)$$ is

called a downwind approximation,

- The backward approximation

$$q\,u'(x) \approx q\left(\frac{u(x) - u(x-h)}{h}\right)$$ is

called an upwind approximation.

(ii) If $q < 0$

- The forward approximation is called an upwind approximation.

- The backward approximation is called a downwind approximation.