Appendix A
Taylor's Theorem

The essential tool in the development of numerical methods is Taylor's theorem. The reason is simple, Taylor's theorem will enable us to approximate a function with a polynomial, and polynomials are easy to compute (most of the time). To start, we define what it means for a function to be C^n.

Definition A.1. Given a non-negative integer n, and an interval $a < x < b$, stating that $f \in C^n(a,b)$ means that $f(x)$, $f'(x)$, $f''(x)$, $\cdots$, $f^{(n)}(x)$ exist and are continuous functions on the interval $a < x < b$.

Note that this definition does not follow the usual convention for exponents. In particular, $f \in C(a,b)$ and $f \in C^0(a,b)$ are the same statement, which are both different than stating that $f \in C^1(a,b)$. If $f \in C(a,b)$, or equivalently if $f \in C^0(a,b)$, then the function is continuous on the interval. In contrast, $f \in C^1(a,b)$ means that $f(x)$ and $f'(x)$ are continuous on the interval. Also, to state that $f \in C^\infty(a,b)$ means $f(x)$ and all of its derivatives are defined and continuous for $a < x < b$.

We now state Taylor's theorem.

Theorem A.1. *Given a function $f(x)$, assume that $f \in C^{n+1}(x_L, x_R)$. In this case, if x and $x+h$ are points in the interval (x_L, x_R), then*

$$f(x+h) = f(x) + hf'(x) + \frac{1}{2}h^2 f''(x) + \cdots + \frac{1}{n!}h^n f^{(n)}(x) + R_{n+1}, \quad \text{(A.1)}$$

where the remainder is

$$R_{n+1} = \frac{1}{(n+1)!}h^{n+1} f^{(n+1)}(\eta), \quad \text{(A.2)}$$

and η is a point between x and $x+h$.

M.H. Holmes, *Introduction to Scientific Computing and Data Analysis*,
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The result in (A.1) is known as Taylor's theorem with remainder. The mystery point η in (A.2) is not known other than it is somewhere in the given interval.

Writing out the first few cases we have that

$$f(x+h) = f(x) + hf'(\eta),$$
$$f(x+h) = f(x) + hf'(x) + \frac{1}{2}h^2 f''(\eta),$$
$$f(x+h) = f(x) + hf'(x) + \frac{1}{2}h^2 f''(x) + \frac{1}{6}h^3 f'''(\eta).$$

The η's in these formulas are not the same. Usually the exact value of η is not important because the remainder term is dropped when using Taylor's theorem to derive an approximation of a function. Doing this, the above expressions become

$$f(x+h) \approx f(x), \tag{A.3}$$
$$f(x+h) \approx f(x) + hf'(x), \tag{A.4}$$
$$f(x+h) \approx f(x) + hf'(x) + \frac{1}{2}h^2 f''(x). \tag{A.5}$$

As a function of h, (A.3) is a constant approximation, (A.4) is a linear approximation, and (A.5) is a quadratic approximation.

There are various ways to write a Taylor expansion. One is as stated in the above theorem, which is

$$f(x+h) = f(x) + hf'(x) + \frac{1}{2}h^2 f''(x) + \cdots + \frac{1}{n!}h^n f^{(n)}(x) + \cdots .$$

The assumption here is that h is close to zero. Another way to write the expansion is as

$$f(x) = f(a) + (x-a)f'(a) + \frac{1}{2}(x-a)^2 f''(a) + \cdots + \frac{1}{n!}(x-a)^n f^{(n)}(a) + \cdots .$$

In this case it is assumed that x is close to a. This gives rise to the linear approximation

$$f(x) \approx f(a) + (x-a)f'(a), \tag{A.6}$$

the quadratic approximation

$$f(x) \approx f(a) + (x-a)f'(a) + \frac{1}{2}(x-a)^2 f''(a), \tag{A.7}$$

and the cubic approximation

$$f(x) \approx f(a) + (x-a)f'(a) + \frac{1}{2}(x-a)^2 f''(a) + \frac{1}{6}(x-a)^3 f'''(a). \tag{A.8}$$

It's certainly possible to write down higher-order approximations, but they are not needed in this text.

A.1 Useful Taylor Series for x Near Zero

$$f(x) = f(0) + xf'(0) + \frac{1}{2}x^2 f''(0) + \frac{1}{6}x^3 f'''(0) + \cdots$$

Power Functions

$$\begin{aligned}
(a+x)^\gamma &= a^\gamma + \gamma x a^{\gamma-1} + \frac{1}{2}\gamma(\gamma-1)x^2 a^{\gamma-2} + \frac{1}{6}\gamma(\gamma-1)(\gamma-2)x^3 a^{\gamma-3} + \cdots \\
\frac{1}{1-x} &= 1 + x + x^2 + x^3 + \cdots \\
\frac{1}{(1-x)^2} &= 1 + 2x + 3x^2 + 4x^3 + \cdots \\
\sqrt{1+x} &= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \cdots \\
\frac{1}{\sqrt{1-x}} &= 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3 + \cdots
\end{aligned}$$

Trig Functions

$$\begin{aligned}
\sin(x) &= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots \\
\arcsin(x) &= x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \cdots \\
\cos(x) &= 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 + \cdots \\
\arccos(x) &= \frac{\pi}{2} - x - \frac{1}{6}x^3 - \frac{3}{40}x^5 + \cdots \\
\tan(x) &= x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots \\
\arctan(x) &= x - \frac{1}{3}x^3 + \frac{1}{40}x^5 + \cdots \\
\cot(x) &= \frac{1}{x} - \frac{1}{3}x - \frac{1}{45}x^3 + \cdots \\
\mathrm{arccot}(x) &= \frac{\pi}{2} - x + \frac{1}{3}x^3 - \frac{1}{5}x^5 + \cdots \\
\sin(a+x) &= \sin(a) + x\cos(a) - \frac{1}{2}x^2\sin(a) + \cdots \\
\cos(a+x) &= \cos(a) - x\sin(a) - \frac{1}{2}x^2\cos(a) + \cdots \\
\tan(a+x) &= \tan(a) + x\sec^2(a) + x^2\tan(a)\sec^2(a) + \cdots
\end{aligned}$$

Exponential and Log Functions

$$
\begin{aligned}
e^x &= 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots \\
a^x = e^{x\ln(a)} &= 1 + x\ln(a) + \frac{1}{2}[x\ln(a)]^2 + \frac{1}{6}[x\ln(a)]^3 + \cdots \\
\ln(a+x) &= \ln(a) + \frac{x}{a} - \frac{1}{2}\left(\frac{x}{a}\right)^2 + \frac{1}{3}\left(\frac{x}{a}\right)^3 + \cdots
\end{aligned}
$$

Hyperbolic Functions

$$
\begin{aligned}
\sinh(x) &= x + \frac{1}{6}x^3 + \frac{1}{120}x^5 + \cdots \\
\operatorname{arcsinh}(x) &= x - \frac{1}{6}x^3 + \frac{3}{40}x^5 + \cdots \\
\cosh(x) &= 1 + \frac{1}{2}x^2 + \frac{1}{24}x^4 + \cdots \\
\operatorname{arccosh}(x) &= \sqrt{2x}\left(1 - \frac{1}{12}x + \frac{3}{160}x^2 + \cdots\right) \\
\tanh(x) &= x - \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots \\
\operatorname{arctanh}(x) &= x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \cdots
\end{aligned}
$$

A.2 Order Symbol and Truncation Error

As a typical example of how we will use Taylor's theorem, for h close to zero

$$
\sin(h) = h - \frac{1}{3!}h^3 + \frac{1}{5!}h^5 - \frac{1}{7!}h^7 + \cdots .
$$

From this we have the approximations

$$
\sin(h) \approx h,
$$

and

$$
\sin(h) \approx h - \frac{1}{3!}h^3.
$$

It is useful to have a way to indicate how the next term in the series depends on h. The big-O notation is used for this, and we write

$$
\sin(h) = h + O(h^3), \tag{A.9}
$$

and
$$\sin(h) = h - \frac{1}{3!}h^3 + O(h^5). \tag{A.10}$$

In this text, the part of the series that is dropped when deriving an approximation is often designated as τ. Given where it comes from, τ is referred to as the *truncation error*. Using the above example, we will sometimes write (A.9) as
$$\sin(h) = h + \tau,$$
where $\tau = O(h^3)$. Similarly, (A.10) can be written as
$$\sin(h) = h - \frac{1}{3!}h^3 + \tau,$$
where $\tau = O(h^5)$.

The definition for big-O is given below. There are more general definitions, but they are not needed here.

Definition A.2. For h close to zero, $\tau = O(h^n)$ means that
$$\lim_{h\to 0} \frac{\tau}{h^n} = L,$$
where $-\infty < L < \infty$.

We will occasionally need to know how big-O terms combine. The rules that cover many of the situations we will come across are the following:

Lemma:

1) If $n < m$, then $O(h^n) + O(h^m) = O(h^n)$.
2) For any nonzero constant α, $O(\alpha h^n) = \alpha O(h^n) = O(h^n)$.

The proof of these statements comes directly from the definition. As an example of how they are used, if $f(h) = 1+2h+O(h^3)$ and $g(h) = -4+3h+O(h^4)$ then
$$\begin{aligned} f + 2g &= 1 + 2h + O(h^3) + 2[-4 + 3h + O(h^4)] \\ &= -7 + 8h + O(h^3) \end{aligned}$$

For the same reason,
$$-2f + 6g = -26 + 14h + O(h^3).$$

The last topic concerns two ways that the truncation error can be written. These come from writing the Taylor series using the remainder term, or else writing it out as a series. For example, one can write the series version
$$\sin(h) = h - \frac{1}{3!}h^3 + \frac{1}{5!}h^5 - \frac{1}{7!}h^7 + \cdots ,$$

as

$$\sin(h) = h + \tau,$$

where

$$\tau = -\frac{1}{3!}h^3 + \frac{1}{5!}h^5 - \frac{1}{7!}h^7 + \cdots . \tag{A.11}$$

In contrast, the remainder form, coming from (A.1) and (A.2), is

$$\sin(h) = h + \tau,$$

where

$$\tau = -\frac{1}{3!}h^3 \cos(\eta). \tag{A.12}$$

In the text, for both cases, the error term is written as $\tau = O(h^3)$. For the series version in (A.11) this should be interpreted as an asymptotic form of the error. What this means is that as h approaches zero, the first term approximation of τ has the stated dependence on h. More explanation about asymptotic forms of an approximation can be found in Holmes [2013].