1. Let $f(x) = \dfrac{5x}{(x^2+3)^2}$. Find its power series about $x=0$, and state the radius of convergence.

Answer

$$\int f(x)dx = 5\int \frac{x}{(x^2+3)^2}\,dx$$

$$= \frac{5}{2}\int \frac{1}{u^2}\,du = -\frac{5}{2u} + C$$

$u = x^2+3$

$du = 2x\,dx$

$$= -\frac{5}{2(x^2+3)} + C.$$

Thus $\quad f(x) = -\dfrac{5}{2}\dfrac{d}{dx}\dfrac{1}{x^2+3}$.

Now $\quad \dfrac{1}{x^2+3} = \dfrac{1}{3-(-x^2)} = \dfrac{1}{3}\cdot\dfrac{1}{1-(-x^2/3)}$

$$= \frac{1}{3}\sum_{n=0}^{\infty}(-x^2/3)^n \quad \text{when} \quad \frac{x^2}{3} < 1.$$

Therefore, for $-\sqrt{3} < x < \sqrt{3}$,

$$f(x) = -\frac{5}{6} \sum_{n=0}^{\infty} \frac{(-1)^n}{3^n} 2n\, x^{2n-1}$$

$$= \sum_{n=1}^{\infty} 5 \frac{(-1)^{n+1} n}{3^{n+1}} x^{2n-1}.$$

The radius of convergence is $\sqrt{3}$.

2. Find the radius and interval of convergence for

$$\sum_{n=0}^{\infty} \frac{(-3)^n (x+4)^n}{\sqrt{n+2}}.$$

Answer

Using the ratio test, we have that

$$\frac{3^{n+1} |x+4|^{n+1}}{\sqrt{n+3}} \cdot \frac{\sqrt{n+2}}{3^n |x+4|^n} = \sqrt{\frac{n+2}{n+3}}\, 3|x+4|$$

$$\xrightarrow[n \to \infty]{} 3|x+4|.$$

Thus, if $3|x+4| < 1$, then the series converges.

Thus the radius of convergence is $1/3$ and the interval of convergence contains $(-4-1/3, -4+1/3)$.

We now check the endpoints separately,

for $x = -4 - 1/3$, we have that

$$\sum_{n=0}^{\infty} \frac{(-3)^n (x+4)^n}{\sqrt{n+2}} = \sum_{n=0}^{\infty} \frac{(-3)^n (-1/3)^n}{\sqrt{n+2}}$$

$$= \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+2}} .$$

This series diverges by the limit comparison test with $\frac{1}{\sqrt{n}}$:

$$\frac{1}{\sqrt{n+2}} \cdot \sqrt{n} = \sqrt{\frac{n}{n+2}} \xrightarrow[n \to \infty]{} 1$$

and $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \infty$ since it is a p-series with $p < 1$.

Now when $x = -4 + 1/3$,

$$\sum_{n=0}^{\infty} \frac{(-3)^n (x+4)^n}{\sqrt{n+2}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+2}} .$$

This series converges by the alternating series test:

$$\frac{1}{\sqrt{n+3}} < \frac{1}{\sqrt{n+2}} \quad \text{and}$$

$$\lim_{n \to \infty} \frac{1}{\sqrt{n+2}} = 0 .$$

Thus the interval of convergence is

$$[-4 - 1/3 \ , \ -4 + 1/3] \quad \text{and}$$

the radius of convergence is $1/3$. $\square$