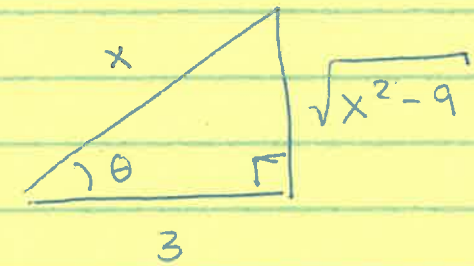
Section 7.3 Problems

13 ~~Find~~ Compute the integral $\int \frac{\sqrt{x^2-9}}{x^3}\,dx$.

Answer:

We use trig substitution



From the right triangle above,

$$\cos\theta = \frac{3}{x} \longrightarrow -\sin\theta\,d\theta = -\frac{3}{x^2}\,dx$$

$$\sin\theta = \frac{\sqrt{x^2-9}}{x}$$

So $\frac{1}{3}\sin^2\theta\,d\theta = \frac{\sqrt{x^2-9}}{x^3}\,dx$.

Making this substitution gives us

$$\int \frac{\sqrt{x^2-9}}{x^3}\,dx = \frac{1}{3}\int \sin^2\theta\,d\theta.$$

Now we recall the identity

$$\sin^2\theta = \frac{1-\cos(2\theta)}{2},$$ so that

$$\frac{1}{3}\int \sin^2\theta \, d\theta = \frac{1}{6}\int 1-\cos(2\theta)\, d\theta$$

$$= \frac{1}{6}\left(\theta - \frac{1}{2}\sin(2\theta)\right) + C$$

Now we substitute back for x.

Since $\cos\theta = \frac{3}{x}$, $\theta = \cos^{-1}\left(\frac{3}{x}\right)$.

Therefore,

$$\int \frac{\sqrt{x^2-9}}{x^3}\, dx = \frac{1}{6}\left(\cos^{-1}\left(\frac{3}{x}\right) - \frac{1}{2}\sin\left(2\cos^{-1}\left(\frac{3}{x}\right)\right)\right)$$

$$+ C. \qquad \square$$

Bonus: If you want to simplify the answer a little bit, you can use the identity

$$\sin(2\theta) = 2\sin\theta\cos\theta = \frac{6\sqrt{x^2-9}}{x^2}, \text{ so that}$$

$$\int \frac{\sqrt{x^2-9}}{x^3}\, dx = \frac{1}{6}\left(\cos^{-1}\left(\frac{3}{x}\right) - \frac{3\sqrt{x^2-9}}{x}\right) + C.$$

$\square$

23 Compute the integral $\int \frac{1}{\sqrt{x^2+2x+5}}\,dx$.

<u>Answer:</u>

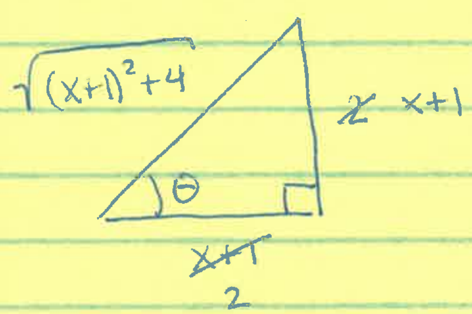
We first complete the square : find a, b st

$$x^2+2x+5 = (x+a)^2 + b^2$$
$$= x^2 + 2ax + a^2 + b^2$$

If we set $a=1$ and $b^2=4$, then
$x^2+2x+5 = (x+1)^2+4$, so

$$\int \frac{1}{\sqrt{x^2+2x+5}}\,dx = \int \frac{1}{\sqrt{(x+1)^2+4}}\,dx$$

Now we use trig substitution



From the right triangle, we have that

$$\tan\theta = \frac{x+1}{2} \longrightarrow \sec^2\theta\, d\theta = \frac{1}{2}dx$$

$$\frac{2}{\sqrt{(x+1)^2+4}} = \cancel{\csc\theta}\ \cos\theta$$

so that $$\frac{1}{\sqrt{(x+1)^2+4}}\,dx = \cancel{\sec^2\theta\csc\theta\, d\theta}\ \sec^2\theta\cos\theta\, d\theta$$

$$= \sec\theta\, d\theta$$

Therefore,

$$\int \frac{1}{\sqrt{(x+1)^2+4}}\,dx = \int \sec\theta \, d\theta$$

$$= \ln\left|\sec\theta + \tan\theta\right| + C$$

~~Now we recall~~

~~Now we substitute back for x~~

Thus, to summarize,

$$\int \frac{1}{\sqrt{x^2+2x+5}}\,dx = \int \frac{1}{\sqrt{(x+1)^2+4}}\,dx$$

$$= \int \sec\theta \, d\theta$$

$$= \ln\left|\frac{\sqrt{(x+1)^2+4}}{2} + \frac{x+1}{2}\right| + C .$$

$\square$