Exam 1 practice solutions

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1. Consider the fixed-point iteration for
$$12x = x^7 - x - 1.$$
For which of the two intervals below will the method converge for any initial value x_0 in that interval? For the intervals that give a convergent method, estimate the number of iterations necessary to obtain approximations accurate to within 10^{-5}, and perform the calculations.

   (a) $[-0.5, 1]$
   (b) $[0.5, 4.2]$

Solution. See Theorem 2.4 in the textbook. The fixed-point iteration is
$$12x_{n+1} = x_n^7 - x_n - 1.$$
Let $g(x) = (x^7 - x - 1)/12$, so that the iteration is in the form $x_{n+1} = g(x_n)$. Then, $g'(x) = (7x^6 - 1)/12$. We have that
$$|g'(x)| < 1 \iff |7x^6 - 1| < 12 \iff -12 < 7x^6 - 1 < 12 \iff |x| < (13/7)^{1/6}.$$
Therefore, from Theorem 2.4, g has a unique fixed point, and any initial condition from interval (a) gives a convergent method, but not from interval (b).

On interval (a), if $-0.5 \leq x \leq 1$, then
$$-0.5 < \frac{(-0.5)^7 - 2}{12} \leq g(x) \leq \frac{0.5}{12} < 1.$$
Therefore, by the same theorem, the fixed point p must be in the interval (a) itself, and the error after the nth iteration is bounded as
$$|x_n - p| \leq k^n \max\{x_0 - a, b - x_0\},$$
where $a = -0.5$, $b = 1$, $k := \max_{x \in [a,b]} |g'(x)|$, and x_0 is the initial condition from $[a, b]$. See Corollary 2.5 in the textbook.

By examining $g'(x)$ on $[a, b]$, we see that $k = 0.5$. The optimal initial condition is $x_0 = (a+b)/2 = 0.25$. Therefore, for this problem, the error is bounded as
$$|x_n - p| \leq 0.25(0.5)^n = (0.5)^{n+2}.$$
To obtain an error within 10^{-5}, we estimate at least 15 iterations. Thankfully, the actual iterations required is only 6.

0	0.250000
1	-0.104161
2	-0.074653
3	-0.077112
4	-0.076907
5	-0.076924
6	-0.076922

□

2. Solve

$$2x - \sin(x) = 1$$

using 3 iterations of Newton's method with initial guess $x_0 = -1$.

Solution. Let $f(x) = 2x - \sin(x) - 1$. Then, Newton's method reads

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{2x_n - \sin(x_n) - 1}{2 - \cos(x_n)}.$$

n	x_n
0	-1
1	0.478750
2	0.931066
3	0.888390

□

3. Consider the equation

$$x = x^3 - 6,$$

with fixed-point solution $p = 2$.

(a) Perform 3 fixed-point iterations with $x_0 = 3$. Why does the method not converge?

Solution. The method is $x_{n+1} = g(x_n)$ with $g(x) = x^3 - 6$. $x_1 = 27 - 6 = 21$. $x_2 = 21^3 - 6 \approx 21^3$. $x_3 = x_2^3 - 6 \approx 21^9 - 6 \approx 21^9$. The method fails to converge because the derivative $g'(x) = 3x^2$ is larger than 1 at the solution $x = p$. □

(b) Now, perform 3 iterations of the related Newton method for this problem with $x_0 = 3$. Does the method converge?

Solution. The method is $x_{n+1} = x_n - f(x_n)/f'(x_n)$, where $f(x) = x - x^3 + 6$.

n	x_n
0	3.000000
1	2.307692
2	2.041819
3	2.000924

The method converges. □

4. Let $x_0 = 0, x_1 = 1, x_2 = 1.5$.

   (a) Construct the Lagrange interpolating polynomials of minimal degree with respect to these points.

   Solution. The general form of a Lagrange interpolating polynomial is

$$L_i(x) = \prod_{j \neq i} \frac{x - x_j}{x_i - x_j}.$$

   Applied to these points:

$$\begin{aligned} L_0(x) &= \frac{(x-1)(x-1.5)}{(0-1)(0-1.5)}, \\ L_1(x) &= \frac{(x-0)(x-1.5)}{(1-0)(1-1.5)}, \\ L_2(x) &= \frac{(x-1)(x-0)}{(1.5-1)(1.5-0)}. \end{aligned}$$

   □

   (b) Find using the previous part, find the interpolation of $f(x) = x^7$.

   Solution. To interpolate a function $f(x)$:

$$p(x) = \sum_i f(x_i) L_i(x).$$

   For $f(x) = x^7$:

$$p(x) = 0^7 L_0(x) + 1^7 L_1(x) + (1.5)^7 L_2(x).$$

   □

   (c) Using the previous part, estimate $f(0.5)$.

   Solution. We estimate with the interpolating polynomial:

$$\begin{aligned} L_1(0.5) &= \frac{(0.5-0)(0.5-1.5)}{(1-0)(1-1.5)} = \frac{0.5(-1)}{-0.5} = 1, \\ L_2(0.5) &= \frac{(0.5-0)(0.5-1)}{(1.5-0)(1.5-1)} = \frac{0.5(-0.5)}{(1.5)(0.5)} = -\frac{1}{3}, \\ p(0.5) &= L_1(0.5) + (1.5)^7 L_2(0.5) = 1 + (1.5)^7/3. \end{aligned}$$

   □

(d) Find an upper bound for the maximum error between $f(x)$ and $p(x)$ on $[0, 1.5]$.

Solution. We recall the error representation formula:

$$f(x) - p(x) = \frac{f^{(n+1)}(\xi(x))}{(n+1)!} \prod_{i=0}^{n} (x - x_i),$$

where $\xi(x) \in [0, 1.5]$. For this problem:

$$\begin{aligned} f(x) - p(x) &= \frac{7(6)(5)\xi(x)^4}{6}(x-0)(x-1)(x-1.5) \\ &= \frac{35}{2}\xi(x)^4 x(x-1)(2x-3) \\ &= \frac{35}{2}\xi(x)^4(2x^3 - 5x^2 + 3x). \end{aligned}$$

Since $0 \le \xi(x) \le 1.5$, $\xi(x)^4 \le (1.5)^4$. Let $q(x) = 2x^3 - 5x^2 + 3x$. Then, $q'(x) = 6x^2 - 10x + 3 = 0$ when

$$x = \frac{10 \pm \sqrt{100 - 4(6)(3)}}{12} = \frac{5 \pm \sqrt{7}}{6} = x_0, x_1,$$

where $x_0 = \frac{5-\sqrt{7}}{6}$ and $x_1 = \frac{5+\sqrt{7}}{6}$. Inserting these into $q(x)$:

$$\begin{aligned} q(x_0) &= 2\frac{(5-\sqrt{7})^3}{6^3} - 5\frac{(5-\sqrt{7})^2}{36} + 3\frac{5-\sqrt{7}}{6} \approx 0.264077, \\ q(x_1) &= 2\frac{(5+\sqrt{7})^3}{6^3} - 5\frac{(5+\sqrt{7})^2}{36} + 3\frac{5+\sqrt{7}}{6} \approx -0.078892. \end{aligned}$$

Therefore,

$$\max_{x \in [0,1.5]} |f(x) - p(x)| \le \frac{35}{2}(1.5)^4(0.264077) \approx 46.791052.$$

$\square$

5. (a) Find the Hermite interpolation to $f(x) = x^4 - x + 1$ using points $x_0 = 1$, $x_1 = 2$. Use divided differences.

Solution. We compute

$$f'(x) = 4x^3 - 1.$$

Therefore,

$$f(1) = 1,\ f'(1) = 3,\ f(2) = 15,\ f'(2) = 31.$$

We use this data in the divided difference algorithm as follows. First, we set $z_0 = z_1 = x_0 = 1$, $z_2 = z_3 = x_1 = 2$. As notation, we let $f_i = f(z_i)$, $f_{i,j} = f[z_i, z_j]$, etc. Then

$$\begin{array}{lll}
f_{0,1} = f'(1) = 3 & f_{0,1,2} = \dfrac{f_{1,2} - f_{0,1}}{z_2 - z_0} = 11 & f_{0,1,2,3} = \dfrac{f_{1,2,3} - f_{0,1,2}}{z_3 - z_0} = 6 \\
f_{1,2} = \dfrac{f_2 - f_1}{z_2 - z_1} = 14 & f_{1,2,3} = \dfrac{f_{2,3} - f_{1,2}}{z_3 - z_1} = 17 & \\
f_{2,3} = f'(2) = 31 & &
\end{array}$$

Therefore, the Hermite interpolating polynomial is

$$\begin{aligned}
p(x) &= f_0 + f_{0,1}(x - z_0) + f_{0,1,2}(x - z_0)(x - z_1) + f_{0,1,2,3}(x - z_0)(x - z_1)(x - z_2) \\
&= 1 + 3(x - 1) + 11(x - 1)^2 + 6(x - 1)^2(x - 2).
\end{aligned}$$

□

(b) Using the previous part, estimate $f(1.5)$.

Solution.

$$f(1.5) \approx p(1.5) = 9/2.$$

□

6. A natural cubic spline on $[0,2]$ is defined by

$$S(x) = \begin{cases} S_0(x) := a + bx + cx^3 & 0 \le x \le 1, \\ S_1(x) := 2 + d(x-1) + e(x-1)^2 + f(x-1)^3 & 1 \le x \le 2. \end{cases}$$

Find the coefficients a, b, c, d, e, and f.

Solution. A natural cubic spline is twice continuously differentiable on $[0,2]$ and has vanishing second derivatives at $x=0$ and $x=2$. This implies that

$$\begin{aligned} S_0(1) &= S_1(1), \\ S_0'(1) &= S_1'(1), \\ S_0''(1) &= S_1''(1), \\ S_0''(0) &= 0, \\ S_1''(2) &= 0. \end{aligned}$$

Substituting for the coefficients:

$$\begin{aligned} a + b + c &= 2, \\ b + 3c &= d, \\ 6c &= 2e, \\ 2e + 6f &= 0. \end{aligned}$$

This system is underdetermined: there are more unknowns than equations. The best we can do is reduce the system to as few unknowns as possible. Writing in matrix-vector form:

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & -1 & 0 & 0 \\ 0 & 0 & 6 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 2 & 6 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Since this matrix is in echelon form, the nonzero entries of each row imply that we can solve for a, b, c, e in terms of d and f:

$$\begin{aligned} e &= -3f, \\ c &= e/3 = -f, \\ b &= d - 3c = d + f, \\ a &= 2 - b - c = 2 - d. \end{aligned}$$

Thus, the reduced form of the cubic spline is

$$S(x) = \begin{cases} 2 - d + (d+f)x - fx^3 & 0 \le x \le 1, \\ 2 + d(x-1) - 3f(x-1)^2 + f(x-1)^3 & 1 \le x \le 2. \end{cases}$$

□