Exam 2 Practice Solutions

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1. (a) Compute the derivative of $f(x) = \log(x)$ at $x_0 = 1$ with $h = 0.1$ and $h = 0.05$ using

$$f'(x) \approx \frac{f(x+4h) - f(x-2h)}{6h}.$$

Solution. For $h = 0.1$,

$$f'(1) \approx \frac{\log(1.4) - \log(0.8)}{0.6} \approx 0.9326929798923711.$$

For $h = 0.05$,

$$f'(1) \approx \frac{\log(1.2) - \log(0.9)}{0.3} \approx 0.9589402415059362.$$

□

(b) Derive the error of the method using Taylor expansions.

Solution.

$$f(x+4h) = f(x) + 4hf'(x) + 8h^2 f''(\xi_+)$$

for some $\xi_+ \in (x, x+4h)$, and

$$f(x-2h) = f(x) - 2hf'(x) + 2h^2 f''(\xi_-)$$

for some $\xi_- \in (x-2h, x)$. Therefore,

$$\frac{f(x+4h) - f(x-2h)}{6h} - f'(x) = \frac{h}{3}(f''(\xi_+) - f''(\xi_-)) + hf''(\xi_+).$$

Using the Mean Value Theorem, there is $\xi \in (\xi_-, \xi_+)$ such that

$$f''(\xi_+) - f''(\xi_-) = f'''(\xi)(\xi_+ - \xi_-).$$

Thus, since $\xi_+ - \xi_- \leq x + 4h - (x-2h) = 6h$,

$$\left| \frac{f(x+4h) - f(x-2h)}{6h} - f'(x) \right| \leq 2h^2 \max_{\xi \in [x-2h, x+4h]} |f'''(\xi)| + h \max_{\xi \in [x, x+4h]} |f''(\xi)|.$$

For $f(x) = \log(x)$,

$$\left| \frac{f(x+4h) - f(x-2h)}{6h} - f'(x) \right| \leq \frac{4h^2}{(x-2h)^3} + \frac{h}{x^2} = \mathcal{O}\left(\frac{h}{x^2}\right).$$

□

2. Approximate

$$\int_{-3}^{3} x^2 - x\,dx$$

by using Lagrange interpolation with quadrature points $x = -2$ and $x = 2$.

Solution. Let $f(x) = x^2 - x$, let $x_0 = -2$, $x_1 = 2$, and let

$$\begin{aligned} L_0(x) &= \frac{x - x_1}{x_0 - x_1} = -\frac{x-2}{4}, \\ L_1(x) &= \frac{x - x_0}{x_1 - x_0} = \frac{x+2}{4}, \\ p(x) &= f(x_0)L_0(x) + f(x_1)L_1(x) = -6\frac{x-2}{4} + 2\frac{x+2}{4}. \end{aligned}$$

Then,

$$\begin{aligned} \int_{-3}^{3} x^2 - x\,dx \approx \int_{-3}^{3} p(x)\,dx &= f(x_0)\int_{-3}^{3} L_0(x)\,dx + f(x_1)\int_{-3}^{3} L_1(x)\,dx \\ &= 6 \cdot 3 + 2 \cdot 3 = 24. \end{aligned}$$

□

3. Find the coefficients a, b, c, d such that

$$\int_{-1}^{1} f(x)\,dx \approx af(-1) + bf(0) + cf'(-1) + df'(0)$$

has degree of precision 3.

Solution. Using the monomial basis $f(x) = 1,\ f(x) = x,\ f(x) = x^2,\ f(x) = x^3$,

$$\begin{aligned} 2 &= a + b, \\ 0 &= -a + c + d, \\ 2/3 &= a - 2c, \\ 0 &= -a + 3c. \end{aligned}$$

We solve the last 2 equations for a and c. Adding:

$$2/3 = c.$$

Then, $a = 3c = 2$. Then, from the first equation, $b = 0$, and, from the second equation, $d = a - c = 2 - 2/3 = 4/3$. Thus,

$$a = 2,\ b = 0,\ c = 2/3,\ d = 4/3.$$

□

4. Solve $y' = -2y + 2t$, $y(0) = 0$ using the forward and backward Euler methods with timestep $\Delta t = 0.2$ to final time $T = 1$. Discuss your results.

Solution. Forward Euler:

$$y_{n+1} = y_n + 2(\Delta t)(t_n - y_n).$$

Backward Euler:

$$y_{n+1} = y_n + 2(\Delta t)(t_{n+1} - y_{n+1}).$$

Solving for y_{n+1}:

$$y_{n+1} = \frac{y_n + 2(\Delta t)(t_n + \Delta t)}{1 + 2\Delta t}.$$

Starting with $y_0 = 0$,

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FE
0.0 0.00000
0.2 0.00000
0.4 0.08000
0.6 0.20800
0.8 0.36480
1.0 0.53888

BE
0.0 0.00000
0.2 0.05714
0.4 0.15510
0.6 0.28221
0.8 0.43015
1.0 0.59296
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The forward Euler solution is always less than the backward Euler solution, and the forward Euler solution always gives 0 as the solution for the first time step. It appears that the backward Euler method is better for this problem with this choice of timestep. □

5. Consider the following multistep method:

$$w_{i+1} = \frac{1}{2}w_i + \frac{1}{2}w_{i-1} + \frac{h}{2}f(t_i, w_{i+1}) + \frac{h}{2}f(t_i, w_i).$$

(a) Verify that the method is zero-stable.

Solution. The characteristic polynomial for this method is

$$p(\lambda) = \lambda^2 - \frac{1}{2}\lambda - \frac{1}{2}.$$

This has roots

$$\lambda = \frac{1/2 \pm \sqrt{1/4 + 4/2}}{2} = \frac{1}{4} \pm \frac{3}{4} = -\frac{1}{2},\ 1.$$

Since all roots have magnitude at most 1, and since the $\lambda = 1$ root has multiplicity 1, the method is strongly zero-stable. □

(b) Derive the condition on h required for linear stability.

Solution. Inserting $f(t, w) = \lambda w$ with $\lambda < 0$,

$$w_{i+1} = \frac{1}{2}w_i + \frac{1}{2}w_{i-1} + \frac{h}{2}\lambda w_{i+1} + \frac{h}{2}\lambda w_i.$$

Then, moving everything to one side and collecting terms:

$$w_{i+1}(1 - h\lambda/2) - w_i(1 + h\lambda)/2 - 1/2 = 0.$$

The corresponding characteristic polynomial is

$$p(z) = (1 - h\lambda/2)z^2 - (1/2 + h\lambda/2)z - 1/2.$$

Linear stability requires that the roots of this polynomial must be less than 1 in magnitude. Since the roots of p don't change if we multiply by a constant, let's multiply by 2 for convenience and look at the roots of

$$q(z) = (2 - h\lambda)z^2 - (1 + h\lambda)z - 1.$$

The roots are

$$z = \frac{1 + h\lambda \pm \sqrt{(1 + h\lambda)^2 + 4(2 - h\lambda)}}{4 - 2h\lambda}.$$

Thus, the linear stability condition is

$$\left| \frac{1 + h\lambda \pm \sqrt{(1 + h\lambda)^2 + 4(2 - h\lambda)}}{4 - 2h\lambda} \right| < 1.$$

□