Exam 3 Practice Solutions

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1. (a) Write down the Jacobi and Gauß–Seidel iterative methods for

$$Ax = b$$

with

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Solution. Decompose A into

$$L := \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad D := \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad U := \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

which are the strictly lower-triangular, diagonal, and strictly upper-triangular parts of A. Starting with an initial guess x^0, the Jacobi method iteratively solves the system

$$Dx^{n+1} + (L + U)x^n = b.$$

The Gauß–Seidel method iteratively solve the system

$$(L + D)x^{n+1} + Ux^n = b.$$

□

(b) Do 2 iterations of the Gauß–Seidel method starting with $x^0 = (0,\ 0,\ 0)^T$.

Solution. Each iteration requires solving the triangular system

$$\begin{cases} 2x_1^{n+1} = 1 - x_2^n, \\ x_1^{n+1} + 2x_2^{n+1} = 1, \\ x_3^{n+1} = 1. \end{cases} \implies \begin{cases} x_1^{n+1} = (1 - x_2^n)/2, \\ x_2^{n+1} = (1 - x_1^{n+1})/2, \\ x_3^{n+1} = 1. \end{cases}$$

Thus,

$$x^1 = \begin{bmatrix} 1/2 \\ 1/4 \\ 1 \end{bmatrix}, \qquad x^2 = \begin{bmatrix} 3/8 \\ 5/16 \\ 1 \end{bmatrix}.$$

□

(c) For each method, compute the T matrix such that $x^{n+1} = Tx^n + c$ and its matrix norm $\|T\|_\infty$.

Solution. Jacobi:

$$T = -D^{-1}(L + U) = -\begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1/2 & 0 \\ -1/2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus,

$$\|T\|_\infty = \max_i \sum_j |T_{i,j}| = 1/2.$$

Gauß–Seidel:

$$T = -(L+D)^{-1}U = -\begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Computing $(L+D)^{-1}$ via Gaussian elimination:

$$\begin{bmatrix} 2 & 0 & 0 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & -1/4 & 1/2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

Thus,

$$T = -\begin{bmatrix} 1/2 & 0 & 0 \\ -1/4 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1/2 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$\|T\|_\infty = \max_i \sum_j |T_{i,j}| = 1/2.$$

□

2. Perform an LU factorization of the matrix

$$A := \begin{bmatrix} 1 & 0 & 1 \\ 3 & 1 & 1 \\ 6 & 1 & 1 \end{bmatrix}$$

where L is lower-triangular with all 1's on the diagonal and U is upper-triangular.

Solution. We perform Gaussian elimination on A to obtain U. By tracking each elimination step as a multiplication by an elementary matrix, we use the elementary matrices to find L.

$$\begin{bmatrix} 1 & 0 & 1 \\ 3 & 1 & 1 \\ 6 & 1 & 1 \end{bmatrix} \overset{E_1}{\to} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 6 & 1 & 1 \end{bmatrix} \overset{E_2}{\to} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 1 & -5 \end{bmatrix} \overset{E_3}{\to} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & -3 \end{bmatrix} = U.$$

The elementary matrices are

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -6 & 0 & 1 \end{bmatrix},$$

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix},$$

so that the sequence of Gaussian elimination steps is equivalent to

$$E_3 E_2 E_1 A = U.$$

Thus,

$$L = (E_3 E_2 E_1)^{-1} = E_1^{-1} E_2^{-1} E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 6 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 6 & 1 & 1 \end{bmatrix}.$$

□

3. Consider the matrix

$$A = \begin{bmatrix} 1 & a \\ 1/5 & 1 \end{bmatrix},$$

where a is a real number.

(a) For which values of a is the matrix singular?

Solution.

$$\det A = 1 - a/5 = 0$$

when $a = 5$. □

(b) Write the Jacobi iterative method to solve $Ax = b$ with the matrix A above and $b = [1,\ 1]^T$. For which values of a will the method converge? (Hint: consider the spectral radius $\rho(T) := \max\{|\lambda| : \lambda \text{ eigenvalue of } T\}$ of the T matrix).

Solution. The original hint is insufficient. You cannot use just any matrix norm to determine completely when the iterative method converges. You must compute the spectral radius of T, and the method converges iff $\rho(T) < 1$. See Theorem 7.19 in the textbook and the surrounding discussion.

Using our work from a previous problem, the T matrix for the Jacobi method is

$$T = -D^{-1}(L+U) = \begin{bmatrix} 0 & -a \\ -1/5 & 0 \end{bmatrix}.$$

Its eigenvalues are computed from solving

$$\det(T - \lambda I) = \lambda^2 - a/5 = 0.$$

Thus, $\lambda = \pm\sqrt{a/5}$, so

$$\rho(T) = \sqrt{|a|/5} < 1$$

when $-5 < a < 5$. Thus, when $-5 < a < 5$, the Jacobi method converges. When $|a| \geq 5$, the Jacobi method diverges. □

4. (a) Use Gram–Schmidt to construct an orthogonal basis $\phi_0(x)$, $\phi_1(x)$, $\phi_2(x)$, of quadratic polynomials on $[1, 2]$ starting from 1, x, x^2.

Solution. Let $(f, g) := \int_1^2 f(x)g(x)\,dx$. Then

$$\begin{aligned}
\phi_0(x) &= 1, \\
\phi_1(x) &= x - \frac{(x, \phi_0)}{(\phi_0, \phi_0)}\phi_0(x) = x - \frac{\int_1^2 x\,dx}{\int_1^2 dx} = x - 3/2, \\
\phi_2(x) &= x^2 - \frac{(x^2, \phi_1)}{(\phi_1, \phi_1)}\phi_1(x) - \frac{(x^2, \phi_0)}{(\phi_0, \phi_0)}\phi_0(x) = x^2 - \frac{\int_1^2 x^2(x - 3/2)\,dx}{\int_1^2 (x-3/2)^2\,dx}(x - 3/2) - \int_1^2 x^2\,dx \\
&= x^2 - 3x + 13/6
\end{aligned}$$

□

(b) Use the orthogonal basis constructed above to find the least-squares approximation to $f(x) = x^4 - x$.

Solution. Let $p(x) = a_0\phi_0(x) + a_1\phi_1(x) + a_2\phi_2(x)$ be the least-squares approximation to f expressed in the orthogonal basis. Then, the coefficients a_i are chosen to minimize the error

$$\int_1^2 \left(f(x) - \sum_i a_i\phi_i(x) \right)^2 dx = \int_1^2 f(x)^2\,dx - 2\sum_i a_i \int_1^2 f(x)\phi_i(x)\,dx + \sum_i a_i^2 \int_1^2 \phi_i(x)^2\,dx.$$

The equality holds only because the ϕ_i are orthogonal. Taking partial derivatives of the error above with respect to the a_i and setting to 0, we conclude that the coefficients are

$$a_i = \frac{\int_1^2 f(x)\phi_i(x)\,dx}{\int_1^2 \phi_i(x)^2\,dx}.$$

Thus,

$$\begin{aligned} a_0 &= \frac{\int_1^2 x^4 - x\,dx}{\int_1^2 dx} = 47/10 \\ a_1 &= \frac{\int_1^2 (x^4 - x)(x - 3/2)\,dx}{\int_1^2 (x - 3/2)^2\,dx} = 67/5 \\ a_2 &= \frac{\int_1^2 (x^4 - x)(x^2 - 3x + 13/6)\,dx}{\int_1^2 (x^2 - 3x + 13/6)^2\,dx} = 96/7 \\ p(x) &= \frac{96}{7}x^2 - \frac{971}{35}x + \frac{501}{35}. \end{aligned}$$

I computed everything in this problem exactly using a symbolic calculator, but decimal approximations for the coefficients are also okay. ☐

5. Consider the matrix

$$A = \begin{bmatrix} 2 & 0 \\ 2 & 0 \\ 0 & 2 \end{bmatrix}$$

(a) Find a singular value decomposition $A = U\Sigma V^T$.

Solution. The matrix A has dimensions $m = 3$ by $n = 2$. Therefore, U is a $m \times m$ orthogonal matrix, Σ is a $m \times n$ matrix with the singular values on the main diagonal, and V is a $n \times n$ orthogonal matrix. We let $u_i \in \mathbb{R}^m$ denote the ith column of U, $\sigma_i \geq 0$ denote the ith singular value, and $v_i \in \mathbb{R}^n$ denote the ith column of V.

To find the singular values and V, we consider the $n \times n$ matrix

$$A^T A = \begin{bmatrix} 8 & 0 \\ 0 & 4 \end{bmatrix}.$$

This has eigenvalues $\lambda_1 = 8$, $\lambda_2 = 4$. Therefore, the singular values are $\sigma_1 = \sqrt{8}$, $\sigma_2 = 2$. The columns of V are found via

$$A^T A v_i = \lambda_i v_i$$

and normalizing so $\|v_i\|_2 = 1$. Thus,

$$\begin{cases} 8v_{1,1} = 8v_{1,1} \\ 4v_{2,1} = 8v_{2,1} \\ v_{1,1}^2 + v_{2,1}^2 = 1 \end{cases} \implies v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$\begin{cases} 8v_{1,2} = 4v_{1,2} \\ 4v_{2,2} = 4v_{2,2} \\ v_{1,2}^2 + v_{2,2}^2 = 1 \end{cases} \implies v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

To find the matrix U,

$$u_i = \frac{1}{\sigma_i} A v_i, \qquad i = 1, 2$$

and then we find u_3 that is orthogonal to u_1 and u_2. Thus

$$u_1 = \frac{1}{\sqrt{8}} \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix},$$

$$u_2 = \frac{1}{2} \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix},$$

and $u_3 \in \mathbb{R}^m$ is required to be orthogonal to u_1, u_2 and have norm 1. We arrive at the following system for the components of u_3:

$$\begin{aligned} u_{1,3} + u_{2,3} &= 0, \\ u_{3,3} &= 0, \\ u_{1,3}^2 + u_{2,3}^2 + u_{3,3}^2 &= 1. \end{aligned}$$

We see that

$$u_3 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix}$$

satisfies the conditions. Thus, a SVD of A is

$$A = \begin{bmatrix} 2/\sqrt{8} & 0 & 1/\sqrt{2} \\ 2/\sqrt{8} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{8} & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

□

(b) Verify your decomposition from the previous part by computing the right-hand side:

$$A = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T,$$

where the u_i, σ_i, v_i come from the decomposition.

Solution. The outer-product of a vector $u \in \mathbb{R}^m$ and $v \in \mathbb{R}^n$ is an $m \times n$ matrix uv^T with entries

$$(uv^T)_{i,j} = u_i v_j.$$

Therefore, for the vectors u_i, v_i, and singular values σ_i from the previous part

$$\sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T = \sqrt{8} \begin{bmatrix} 2/\sqrt{8} & 0 \\ 2/\sqrt{8} & 0 \\ 0 & 0 \end{bmatrix} + 2 \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} = A.$$

□

(c) Order the singular values $\sigma_1 \geq \sigma_2$. Compute

$$A_1 := \sigma_1 u_1 v_1^T$$

and

$$\|A - A_1\|_2,$$

where

$$\|A\|_2 := \sqrt{\lambda_{\max}}$$

and $\lambda_{\max}$ is the largest eigenvalue of AA^T or $A^T A$.

Solution. From the previous part,

$$A_1 = \sqrt{8} \begin{bmatrix} 2/\sqrt{8} & 0 \\ 2/\sqrt{8} & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$A - A_1 = 2 \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Since, for any matrix A, AA^T and $A^T A$ have the same eigenvalues, it is convenient to compute

$$(A - A_1)^T (A - A_1) = \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix}.$$

This has eigenvalues $\lambda = 0$ and $\lambda = 4$, so $\|A - A_1\|_2 = 2$. □