Final Exam Practice Solutions

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1. Consider the following system:

$$x - \frac{x^4}{12} - \frac{y}{16} = \frac{1}{10},$$
$$y - \frac{x^2}{12} = \frac{1}{5}.$$

   (a) Write the system as a fixed-point problem and show that there is a unique fixed point in the set

$$D := \{(x, y) \mid 0 \le x, y \le 1\}.$$

Solution. Let

$$F(x, y) := \begin{bmatrix} \frac{x^4}{12} + \frac{y}{16} + \frac{1}{10} \\ \frac{x^2}{12} + \frac{1}{5} \end{bmatrix}.$$

Then solving the system is equivalent to solving the fixed-point problem

$$F(x, y) = (x, y).$$

From Theorem 10.6 in Lecture 17 of the professors notes, if

$$\max_{(x,y)\in D} |\partial_j F_i(x, y)| < 1/2,$$

then there is a unique fixed point in D. Computing for $(x, y) \in D$:

$$\begin{aligned} \partial_1 F_1(x, y) &= \frac{x^3}{3} \in [0, 1/3], \\ \partial_2 F_1(x, y) &= \frac{1}{16} \\ \partial_1 F_2(x, y) &= \frac{x}{6} \in [0, 1/6], \\ \partial_2 F_2(x, y) &= 0, \end{aligned}$$

so the condition is satisfied. This completes the proof. □

   (b) Write Newton's method for this system and perform 1 iteration.

Solution. Let

$$G(x, y) = (x, y) - F(x, y).$$

Then, Newton's method for this system reads

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} x_n \\ y_n \end{bmatrix} - \nabla G(x_n, y_n)^{-1} G(x_n, y_n).$$

Here,

$$\begin{aligned} \nabla G(x, y) &= I - \nabla F(x, y), \\ I &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \\ \nabla F(x, y) &= \begin{bmatrix} \frac{x^3}{3} & \frac{1}{16} \\ \frac{x}{6} & 0 \end{bmatrix}. \end{aligned}$$

Thus,

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} x_n \\ y_n \end{bmatrix} - \begin{bmatrix} 1 - \frac{x_n^3}{3} & -\frac{1}{16} \\ -\frac{x_n}{6} & 1 \end{bmatrix}^{-1} \begin{bmatrix} x_n - \frac{x_n^4}{12} - \frac{y_n}{16} - \frac{1}{10} \\ y_n - \frac{x_n^2}{12} - \frac{1}{5} \end{bmatrix}.$$

Starting with $x_0 = y_0 = 0$,

$$\begin{aligned} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} &= - \begin{bmatrix} 1 & -\frac{1}{16} \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -\frac{1}{10} \\ -\frac{1}{5} \end{bmatrix} \\ &= \begin{bmatrix} 1 & \frac{1}{16} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{10} \\ \frac{1}{5} \end{bmatrix} \\ &= \begin{bmatrix} 1/10 + 1/80 \\ 1/5 \end{bmatrix} \\ &= \begin{bmatrix} 0.1125 \\ 0.2 \end{bmatrix}. \end{aligned}$$

□

2. Consider solving

$$Ax = b$$

with

$$A = \begin{bmatrix} 4 & 1 \\ 1 & z \end{bmatrix}.$$

(a) For which values of z will the Jacobi iterative method converge for this problem?

Solution. Decompose $A = D + L + U$, where

$$\begin{aligned} D &= \begin{bmatrix} 4 & 0 \\ 0 & z \end{bmatrix}, \\ L &= \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \\ U &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

The Jacobi method reads

$$Dx^{n+1} + (L+U)x^n = b.$$

This method converges if and only if the spectral radius of the iteration matrix

$$T := D^{-1}(L+U)$$

is less than 1. Computing:

$$\begin{aligned} D^{-1} &= \begin{bmatrix} 1/4 & 0 \\ 0 & 1/z \end{bmatrix} \\ L + U &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ T &= \begin{bmatrix} 0 & 1/4 \\ 1/z & 0 \end{bmatrix}. \end{aligned}$$

The characteristic polynomial of T is then

$$\det(T - \lambda I) = \lambda^2 - \frac{1}{4z},$$

so the eigenvalues of T are

$$\lambda = \pm \frac{1}{2\sqrt{z}}.$$

Thus,

$$|\lambda| = \frac{1}{2\sqrt{|z|}} < 1 \iff |z| > \frac{1}{4}.$$

Therefore, the method converges iff $|z| > 1/4$. ☐

(b) For which values of z will the Gauss–Seidel iterative method converge for this problem?

Solution. Decompose $A = D + L + U$, where

$$D = \begin{bmatrix} 4 & 0 \\ 0 & z \end{bmatrix}, \quad L = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad U = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

The Gauss–Seidel method reads

$$(D + L)x^{n+1} + Ux^n = b.$$

This method converges if and only if the spectral radius of the iteration matrix

$$T := (D + L)^{-1}U$$

is less than 1. Computing:

$$D + L = \begin{bmatrix} 4 & 0 \\ 1 & z \end{bmatrix}, \quad (D + L)^{-1} = \begin{bmatrix} 1/4 & 0 \\ -1/(4z) & 1/z \end{bmatrix}, \quad T = \begin{bmatrix} 0 & 1/4 \\ 0 & -1/(4z) \end{bmatrix}.$$

The characteristic polynomial of T is then

$$\det(T - \lambda I) = \lambda(\lambda + \frac{1}{4z}),$$

so the eigenvalues of T are

$$\lambda = 0, \ -\frac{1}{4z}.$$

Thus,

$$\max|\lambda| = \frac{1}{4|z|} < 1 \iff |z| > \frac{1}{4}.$$

Therefore, the method converges iff $|z| > 1/4$. ☐

3. Consider the nonlinear system

$$\begin{aligned} x_1 + x_2^2 - 6 &= 0, \\ -x_1 + x_2 - 1 &= 0. \end{aligned}$$

(a) Use the continuation method on this system with initial condition $x_1 = x_2 = 0$. Write down the resulting differential equation.

Solution. Let

$$F(x) := \begin{bmatrix} x_1 + x_2^2 - 6 \\ -x_1 + x_2 - 1 \end{bmatrix}.$$

The continuation method considers the function

$$G(\lambda, x) := \lambda F(x) + (1 - \lambda)(F(x) - F(0)).$$

We suppose that there is a curve $x(\lambda)$ such that

$$G(\lambda, x(\lambda)) = 0$$

for all λ, and we differentiate:

$$0 = \frac{d}{d\lambda} G(\lambda, x(\lambda)) = \partial_\lambda G + \nabla_x G \frac{dx}{d\lambda}.$$

Thus, $x(\lambda)$ satisfies the differential equation

$$\frac{dx}{d\lambda} = -(\nabla_x G)^{-1} \partial_\lambda G$$

with initial condition $x(0) = 0$. For this problem:

$$\partial_\lambda G = F(0) = \begin{bmatrix} -6 \\ -1 \end{bmatrix},$$

$$\nabla_x G = \nabla F = \begin{bmatrix} 1 & 2x_2 \\ -1 & 1 \end{bmatrix}.$$

☐

(b) Solve the differential equation from the previous part using the forward Euler method. Compute 2 iterations with $\tau = 0.5$.

Solution. Euler's method for this system and with the given timestep size reads

$$x^{(n+1)} = x^{(n)} + 0.5 \begin{bmatrix} 1 & 2x_2^{(n)} \\ -1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 6 \\ 1 \end{bmatrix}$$

Starting with $x^{(0)} = 0$,

$$x^{(1)} = 0.5 \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3.5 \end{bmatrix}$$

$$x^{(2)} = 0.5 \begin{bmatrix} 1 & 7 \\ -1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 6 \\ 1 \end{bmatrix} = \frac{1}{16} \begin{bmatrix} 1 & -7 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/16 \\ 7/16 \end{bmatrix}$$

☐

4. (a) Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function such that $f(p) = 0$ and $f'(p) \neq 0$. Consider the following iterative method:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} + f(x_n)^2 f'(x_n).$$

Show that if x_0 is sufficiently close to p, then $x_n \to p$.

Solution. Let $G(x) = x - f(x)/f'(x) + f(x)^2 f'(x)$. Then, using the ideas from the proof of the convergence of Newton's method in Lecture 18 of the professors notes, if $G'(p) = 0$, then there is $\delta > 0$ such that when $|x_0 - p| < \delta$, the method converges to p. Using the chain rule:

$$G'(x) = 1 - (f'(x)^2 - f(x)f''(x))/f'(x)^2 + 2f(x)f'(x)^2 + f(x)^2 f''(x).$$

Evaluating at p and using the fact that $f(p) = 0$, we see $G'(p) = 0$. This completes the proof. ☐

(b) Now let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a smooth function such that $f(p) = 0$ and $\nabla f(p) \in \mathbb{R}^{2\times 2}$ is invertible. Let $F(x) := (f_1(x)^2, f_2(x)^2)$. Consider the iterative method

$$x^{(n+1)} = x^{(n)} - (\nabla f(x^{(n)}))^{-1} f(x^{(n)}) + F(x^{(n)}).$$

Show that if $x^{(0)}$ is sufficiently close to p, then $x^{(n)} \to p$.

Solution. We follow the same procedure as the previous part, but the computations are slightly more involved. We show that the function

$$G(x) = x - (\nabla f(x))^{-1} f(x) + F(x)$$

satisfies $\nabla G(p) = 0$. This is sufficient to conclude the result. Using the product rule:

$$\begin{aligned}\nabla G(x) &= I - \nabla((\nabla f)^{-1})(x) f(x) - (\nabla f(x))^{-1} \nabla f(x) + \nabla F(x) \\ &= -\nabla((\nabla f)^{-1})(x) f(x) + \nabla F(x).\end{aligned}$$

From the definition of F,

$$\nabla F(x) = \begin{bmatrix} 2f_1(x)\partial_1 f_1(x) & 2f_1(x)\partial_2 f_1(x) \\ 2f_2(x)\partial_1 f_2(x) & 2f_2(x)\partial_2 f_2(x) \end{bmatrix}.$$

Thus, since $f(p) = 0$, $\nabla F(p) = 0$. It then follows that $\nabla G(p) = 0$, as desired. ☐

5. Consider the following multi-step scheme:

$$w_{i+1} = w_i/4 + 3w_{i-1}/4 + hf(t_{i-1}, w_i)/2 + hf(t_{i-1}, w_{i-1})/2.$$

(a) Verify zero-stability.

Solution. We set $f = 0$ and consider the related characteristic polynomial

$$p(z) = z^2 - z/4 - 3/4.$$

This has roots

$$z = \frac{1}{8} \pm \frac{1}{2}\sqrt{\frac{1}{16} + 3} = \frac{1}{8} \pm \frac{7}{8} = -3/4,\ 1.$$

Thus, since all the roots have magnitude at most 1, the method is zero stable. ☐

(b) Find the condition required for linear stability.

Solution. We set $f(t, w) = \lambda w$ with $\lambda < 0$. Collecting terms:

$$w_{i+1} = (1/4 + h\lambda/2)w_i + (3/4 + h\lambda/2)w_{i-1}.$$

We consider the characteristic polynomial

$$p(z) = z^2 - (1/4 + h\lambda/2)z - (3/4 + h\lambda/2).$$

This has roots

$$z = (1/8 + h\lambda/4) \pm \frac{1}{2}\sqrt{(1/4 + h\lambda/2)^2 + (3 + 2h\lambda)}$$

Therefore, the condition required for linear stability is that

$$\left|(1/8 + h\lambda/4) \pm \frac{1}{2}\sqrt{(1/4 + h\lambda/2)^2 + (3 + 2h\lambda)}\right| < 1.$$

☐

6. Consider the matrix

$$A = \begin{bmatrix} z & 0 \\ 1 & 1 \\ 1 & -1 \end{bmatrix}$$

(a) Determine a singular-value decomposition $U\Sigma V^T$ of A.

Solution. We first compute

$$A^T A = \begin{bmatrix} z^2+2 & 0 \\ 0 & 2 \end{bmatrix}$$

This has eigenvalues z^2+2 and 2. A corresponding orthonormal basis of eigenvectors is $(1,0)^T$ and $(0,1)^T$. Therefore,

$$\Sigma = \begin{bmatrix} \sqrt{z^2+2} & 0 \\ 0 & \sqrt{2} \end{bmatrix}, \qquad V = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Letting v_i denote the ith column of V, we compute the first 2 columns of U as

$$u_1 = \frac{1}{\sigma_1} A v_1 = \frac{1}{\sqrt{z^2+2}} \begin{bmatrix} z \\ 1 \\ 1 \end{bmatrix},$$

$$u_2 = \frac{1}{\sigma_2} A v_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

The third column of U must be orthogonal to the first 2 and a unit vector. Thus,

$$\begin{aligned} z u_{3,1} + u_{3,2} + u_{3,3} &= 0, \\ u_{3,2} - u_{3,3} &= 0, \\ u_{3,1}^2 + u_{3,2}^2 + u_{3,3}^2 &= 1. \end{aligned}$$

This is solved by

$$u_3 = \frac{1}{\sqrt{1+z^2/2}} \begin{bmatrix} 1 \\ -z/2 \\ -z/2 \end{bmatrix}.$$

Thus,

$$U = \begin{bmatrix} z/\sqrt{z^2+2} & 0 & 1/\sqrt{1+z^2/2} \\ 1/\sqrt{z^2+2} & 1/\sqrt{2} & -z/(2\sqrt{1+z^2/2}) \\ 1/\sqrt{z^2+2} & -1/\sqrt{2} & -z/(2\sqrt{1+z^2/2}) \end{bmatrix}.$$

□

(b) Verify that

$$A = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T.$$

Solution.

$$\sigma_1 u_1 v_1^T = \sqrt{z^2+2} \frac{1}{\sqrt{z^2+2}} \begin{bmatrix} z \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} z & 0 \\ 1 & 0 \\ 1 & 0 \end{bmatrix},$$

$$\sigma_2 u_2 v_2^T = \sqrt{2} \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix}.$$

Adding these recovers A as desired. □

(c) Order the singular values $\sigma_1 \geq \sigma_2$. Compute $A_1 = \sigma_1 u_1 v_1^T$ and $\|A - A_1\|_2$, where $\|B\|_2$ is the square-root of the largest eigenvalue of $B^T B$ or BB^T.

Solution. A_1 is already computed above, and $A_2 := A - A_1 = \sigma_2 u_2 v_2^T$ is also already computed above. Computing

$$A_2^T A_2 = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix},$$

its largest eigenvalue is $\lambda = 2$, so $\|A_2\|_2 = \sqrt{2} = \sigma_2$. □

7. Consider using a fixed-point iteration to solve the following system in $[0,1] \times [0,1]$:

$$x - ax^2 - y/10 = 1/10,$$
$$y - x^2/12 = 1/5.$$

Compute the Jacobian of the iteration matrix and use it to determine the values of a for which the fixed-point method converges for any initial data in $[0,1] \times [0,1]$.

Solution. We let $F(x,y) = (ax^2 + y/10 + 1/10,\ x^2/12 + 1/5)$. Then

$$\nabla F(x,y) = \begin{bmatrix} 2ax & 1/10 \\ x/6 & 0 \end{bmatrix}.$$

The iterative method converges iff the spectral radius of this matrix is less than 1 for all $0 \leq x, y \leq 1$. Computing the characteristic polynomial:

$$\det(\nabla F(x,y) - \lambda I) = (2ax - \lambda)(-\lambda) - x/60 = \lambda^2 - 2ax\lambda - x/60 = 0$$

when

$$\lambda = ax \pm \sqrt{a^2x^2 + x/60}.$$

The precise set of a for which the method converges is then

$$\{a \mid \max_{x\in[0,1]} |ax \pm \sqrt{a^2x^2 + x/60}| < 1\}.$$

A less sharp, but practical, sufficient bound can be found by the following inqualities:

$$|ax \pm \sqrt{a^2x^2 + x/60}| \leq 2|a|x + \sqrt{x/60} \leq 2|a| + 1/\sqrt{60}.$$

Thus, if we require that

$$2|a| + 1/\sqrt{60} < 1 \iff |a| < \frac{1}{2}(1 - 1/\sqrt{60}) \approx 0.436,$$

this is sufficient to guarantee convergence. □

8. Consider a smooth function $f(x)$ such that

$$f(0) = 1,\ f'(0) = 5,\ f(2) = 32,\ f'(2) = 80.$$

(a) Find the Hermite interpolating polynomial to f using the given data and divided differences.

Solution. The general divided differences table is

$$\begin{array}{lllll} z_0 & f(z_0) & f[z_0,z_1] & f[z_0,z_1,z_2] & f[z_0,z_1,z_2,z_3] \\ z_1 & f(z_1) & f[z_1,z_2] & f[z_1,z_2,z_3] & \\ z_2 & f(z_2) & f[z_2,z_3] & & \\ z_3 & f(z_3) & & & \end{array}$$

where

$$f[z_i,z_j] = (f(z_j) - f(z_i))/(z_j - z_i),$$
$$f[z_i,z_j,z_k] = (f[z_j,z_k] - f[z_i,z_j])/(z_k - z_i),$$
$$f[z_0,z_1,z_2,z_3] = (f[z_1,z_2,z_3] - f[z_0,z_1,z_2])/(z_3 - z_0).$$

In the event that $z_j = z_i$ when computing $f[z_i, z_j]$, we replace the entry with $f'(z_i)$. The interpolating polynomial is then

$$p(x) = f(z_0) + f[z_0, z_1](x - z_0) + f[z_0, z_1, z_2](x - z_0)(x - z_1) + f[z_0, z_1, z_2, z_3](x - z_0)(x - z_1)(x - z_2).$$

For this problem, $z_0 = z_1 = 0$, $z_2 = z_3 = 2$. Thus, the table fills in as

$$\begin{array}{cccccc} 0 & 1 & 5 & (31/2 - 5)/2 & (80 - 31 + 5)/4 \\ 0 & 1 & 31/2 & (80 - 31/2)/2 & \\ 2 & 32 & 80 & & \\ 2 & 32 & & & \end{array},$$

so

$$p(x) = 1 + 5x + \frac{21}{4}x^2 + \frac{54}{4}x^2(x - 2).$$

$\square$

(b) Estimate $f(0.5)$ using the Hermite interpolating polynomial.

Solution.

$$p(0.5) = 1 + 5/2 + \frac{21}{4}\frac{1}{4} + \frac{54}{4}\frac{1}{4}(1/2 - 2) = -1/4.$$

$\square$

9. Let $x_0 = 0$, $x_1 = 1$, $x_2 = 3$.

(a) Construct the Lagrange basis polynomials using these points.

Solution. The polynomials are of the form

$$L_i(x) = \frac{(x - x_j)(x - x_k)}{(x_i - x_j)(x_i - x_k)}.$$

Thus,

$$\begin{aligned} L_0(x) &= \frac{(x - 1)(x - 3)}{3}, \\ L_1(x) &= \frac{x(3 - x)}{2}, \\ L_2(x) &= \frac{x(x - 1)}{6}. \end{aligned}$$

$\square$

(b) Find the Lagrange interpolation of

$$f(x) = x^5.$$

Solution.

$$p(x) = \sum_i f(x_i)L_i(x) = L_1(x) + 3^5 L_2(x) = \frac{x(3 - x)}{2} + 3^5\frac{x(x - 1)}{6}.$$

$\square$

(c) Estimate $f(2)$ using the Lagrange interpolant.

Solution.

$$f(2) \approx p(2) = 1 + 3^4 = 82.$$

$\square$

(d) Find the maximum error between f and its Lagrange interpolant in the interval $[0, 3]$.

Solution. From a theorem in Lecture 3 of the professors notes,

$$f(x) = p(x) + \frac{1}{6} f^{(3)}(\xi) x(x-1)(x-3)$$

for some $\xi \in (0,3)$ and all $x \in (0,3)$. Since $f(x) = x^5$, $f'(x) = 5x^4$, $f''(x) = 20x^3$, $f^{(3)}(x) = 60x^2$. Thus

$$\frac{1}{6} f^{(3)}(\xi) = 10\xi^2 \in [0, 90].$$

Let $q(x) = x(x-1)(x-3) = x(x^2 - 4x + 3) = x^3 - 4x^2 + 3x$. Then $q'(x) = 3x^2 - 8x + 3 = 0$ when

$$x = \frac{8 \pm \sqrt{64 - 36}}{6} \approx 2.21525,\ 0.451416.$$

Inserting these back into q gives

$$\max_{x \in [0,3]} |q(x)| \approx 2.11261 < 2.2$$

Thus, the error is bounded as

$$\max_{x \in [0,3]} |f(x) - p(x)| \leq 10 \max_{\xi \in [0,3]} \xi^2 \max_{x \in [0,3]} |q(x)| < 90(2.2) = 198.$$

$\square$