MATH 437 Homework 1

1. Let f be a smooth function, and let $p \in \mathbb{R}$ such that $f(p) = 0$ and $f'(p) \neq 0$. Let

$$g(x) := x - \frac{f(x)}{f'(x)} + f(x)^2 f''(x)$$

for all $x \in \mathbb{R}$.

   (1) Show $g(p) = p$.
   (2) Show $g'(p) = 0$.

Solution.

   (1) Since $f(p) = 0$ and $f'(p) \neq 0$,

$$g(p) = p - \frac{f(p)}{f'(p)} + f(p)^2 f''(p) = p.$$

   (2) Applying the quotient rule, product rule, and chain rule,

$$g'(x) = 1 - \frac{f'(x)^2 - f(x)f''(x)}{f'(x)^2} + 2f(x)f'(x)f''(x) + f(x)^2 f^{(3)}(x).$$

Evaluating at $x = p$ and using the fact that $f(p) = 0$ but $f'(p) \neq 0$,

$$g'(p) = 1 - \frac{f'(p)^2}{f'(p)^2} = 0.$$

□

2. Let $g(x) = x^3 - x + 1$ for all $x \in \mathbb{R}$.

   (1) Do 3 iterations of the fixed point iteration starting with $x_0 = 1.1$.
   (2) Does the method converge to the fixed point $p = 1$ with this initial value? Why or why not?

Solution.

   (1) The fixed-point iteration is $x_{n+1} = g(x_n)$. Therefore,

$$x_1 = g(x_0) = 1.231$$
$$x_2 = g(x_1) = 1.634409391$$
$$x_3 = g(x_2) = 3.7315787025190668$$

(2) The method does not converge for this initial condition because $g'(p) = g'(1) = 2 > 1$. See Proposition 1.4 in the notes.

$\square$

3. Determine an interval $[a, b]$ where the fixed point iteration will converge for

$$g(x) = \frac{5}{x^2} + 2.$$

Implement the fixed point iteration and choose a starting value x_0 that gives convergence. Iterate until the residual is within 10^{-5} and report the iterates in a table.

Solution. We invoke Theorem 1.3 in the notes, which states that g has a unique fixed point if there is an interval $[a, b]$ and a constant $k < 1$ where $|g'(x)| \leq k$ on $[a, b]$. Since $g(x) > 0$ for all $x \neq 0$, we only consider $x > 0$. Then, for $x > 0$ and $k < 1$,

$$|g'(x)| = \frac{10}{x^3} \leq k \iff x > \sqrt[3]{\frac{10}{k}}.$$

Thus, for any choice of $k < 1$ and any closed subinterval $[a, b]$ of $(\sqrt[3]{10/k}, \infty)$, the theorem holds and we have convergence when x_0 is from this subinterval. For example, taking $k = 1/2$, $a = 3$, $b = 4$ is sufficient.

We choose $x_0 = 3.5$. An implementation is found in the accompanying source code. The output is below.

iteration	x	residual
0	3.500000e+00	1.091837e+00
1	2.408163e+00	4.540171e-01
2	2.862180e+00	2.518346e-01
3	2.610346e+00	1.234477e-01
4	2.733794e+00	6.477439e-02
5	2.669019e+00	3.286688e-02
6	2.701886e+00	1.697221e-02
7	2.684914e+00	8.686494e-03
8	2.693600e+00	4.466320e-03
9	2.689134e+00	2.291034e-03
10	2.691425e+00	1.176629e-03
11	2.690248e+00	6.039174e-04
12	2.690852e+00	3.100660e-04
13	2.690542e+00	1.591694e-04
14	2.690701e+00	8.171497e-05
15	2.690620e+00	4.194931e-05
16	2.690662e+00	2.153564e-05
17	2.690640e+00	1.105569e-05
18	2.690651e+00	5.675657e-06

$\square$

4. Let $g(x) = x^3 - 6$.

   (1) Let $x_0 = 2.001$ and do 5 iterations of the fixed point algorithm. Does the method converge to the fixed point $p = 2$? Why or why not?

   (2) Let $x_0 = 3$ and do 5 iterations of Newton's method on the function $f(x) = x - g(x)$. Does the method converge to the root $p = 2$?

Solution.

(1) Using the implementation from the previous problem,

iteration	x
0	2.001000e+00
1	2.012006e+00
2	2.144939e+00
3	3.868351e+00
4	5.188655e+01
5	1.396837e+05

We conclude that the fixed point iteration does not converge. Using Proposition 1.4 from the notes, the method does not converge because $g'(2) = 12 > 1$.

(2) We recall that Newton's method is $x_{n+1} = x_n - f(x_n)/f'(x_n)$. For this particular problem,

$$x_{n+1} = x_n - \frac{x_n - (x_n^3 - 6)}{1 - 3x_n^2}.$$

Thus,

iteration	x
0	3.000000e+00
1	2.307692e+00
2	2.041820e+00
3	2.000925e+00
4	2.000000e+00
5	2.000000e+00

We conclude that the method converges to the root.

□

5. Let

$$f(x) = e^x + 2^{-x} + 2\cos x - 6.$$

Use Newton's method to find the root of f on the interval $[1, 2]$ until the residual is within 10^{-5}.

Solution. For this particular problem,

$$x_{n+1} = x_n - \frac{e^{x_n} + 2^{-x_n} + 2\cos x_n - 6}{e^{x_n} - 2^{-x_n}\ln 2 - 2\sin x}.$$

The implementation is found in the accompanying source code. We start with $x_0 = 1.5$. The output is below.

iteration	x	residual
0	1.500000e+00	1.023283e+00
1	1.956490e+00	5.797014e-01
2	1.841533e+00	5.034095e-02
3	1.829506e+00	5.021213e-04
4	1.829384e+00	5.151614e-08

□