MATH 437 Homework 2 (20 points)

1. Let $x_0 = 2$, $x_1 = 3$, $x_2 = 4$, $x_3 = 5$.

   (a) (1 point) Construct the Lagrange polynomials of maximal degree interpolating these points.

   (b) (1 point) Find the Lagrange form of the degree 3 interpolant p_3 of $f(x) = \log(x-1)$ using these points.

   Solution.

   (a)

$$L_0(x) = \frac{(x-3)(x-4)(x-5)}{(2-3)(2-4)(2-5)},$$
$$L_1(x) = \frac{(x-2)(x-4)(x-5)}{(3-2)(3-4)(3-5)},$$
$$L_2(x) = \frac{(x-2)(x-3)(x-5)}{(4-2)(4-3)(4-5)},$$
$$L_3(x) = \frac{(x-2)(x-3)(x-4)}{(5-2)(5-3)(5-4)}.$$

   (b)

$$p_3(x) = \log(2)L_1(x) + \log(3)L_2(x) + \log(4)L_3(x).$$

□

2. (a) (1 point) Using the points x_0, x_1, x_2 and the function f from the previous problem, construct the Newton form of the degree 2 interpolant p_2.

   (b) (2 points) Derive a bound on the absolute error $\max_{x\in[2,4]} |p(x) - f(x)|$.

   Solution.

   (a)

		$f[x_i]$	$f[x_i, x_{i+1}]$	$f[x_0, x_1, x_2]$
x_0	2	0	$\log(2)$	$\frac{\log(3)-2\log(2)}{2}$
x_1	3	$\log(2)$	$\log(3) - \log(2)$	
x_2	4	$\log(3)$		

$$p_2(x) = \log(2)(x-2) + \frac{\log(3) - 2\log(2)}{2}(x-2)(x-3).$$

(b) Any valid upper bound will do, as long as there is some mathematical justification for it. We will discuss 2 different bounds: one that is easy to compute and one that is a little more difficult to compute.

Using Theorem 3.9 from the notes, for all $x \in [2,4]$, there is $\xi_x \in [2,4]$ such that

$$f(x) = p_2(x) + \frac{f^{(3)}(\xi_x)}{6}(x - x_0)(x - x_1)(x - x_2).$$

On the interval $[2,4]$,

$$0 < f^{(3)}(\xi_x) = \frac{2}{(\xi_x - 1)^3} \leq 2.$$

Let $N_3(x) = (x-2)(x-3)(x-4)$. To get an easy upper bound that is not sharp,

$$\max_{x\in[2,4]} |N_3(x)| \leq \max_{x\in[2,4]} |x-2| \max_{x\in[2,4]} (x-3) \max_{x\in[2,4]} |x-4| = 4.$$

Therefore,

$$\max_{x\in[2,4]} |p_2(x) - f(x)| \leq \max_{\xi\in[2,4]} f^{(3)}(\xi) \max_{x\in[2,4]} |N_3(x)| \leq 8.$$

To get a sharper bound that is harder to compute, we improve the bound on $|N_3(x)|$. The extreme values of N_3 on $[2,4]$ occur when $N_3'(x) = 0$. To simplify the computation, we change variables $y = x - 2$:

$$N_3(y) = y(y-1)(y-2) = y(y^2 - 3y + 2) = y^3 - 3y^2 + 2y.$$

Then, $N_3(x) = (x-2)^3 - 3(x-2)^2 + 2(x-2)$, so

$$N_3'(x) = 3(x-2)^2 - 6(x-2) + 2.$$

Therefore, $N'(x) = 0$ when

$$x - 2 = \frac{6 \pm \sqrt{36 - 24}}{6} \iff x = 3 \pm \frac{\sqrt{3}}{3}.$$

Inserting these values of x into $N_3(x)$ gives

$$\max_{x\in[2,4]} |N_3(x)| = N_3\left(3 - \frac{\sqrt{3}}{3}\right) = \frac{2\sqrt{3}}{9}.$$

Thus, a sharper bound is

$$\max_{x\in[2,4]} |p_2(x) - f(x)| \leq \frac{4\sqrt{3}}{9}.$$

$\square$

3. (a) (1 point) Find the Hermite interpolating polynomial p for $f(x) = e^{2x}$ using the data

$$f(0) = 1,\ f'(0) = 2,\ f(1) = e^2,\ f'(1) = 2e^2$$

via divided differences.

(b) (2 points) Using the polynomial p from the previous part, estimate $f(0.5)$.

Solution.

(a) We set $z_0 = z_1 = 0$ and $z_2 = z_3 = 1$. Then

$$\begin{aligned}
f(z_0) &= f(z_1) = 1, \\
f(z_2) &= f(z_3) = e^2, \\
f[z_0, z_1] &= f'(z_0) = 2, \\
f[z_1, z_2] &= \frac{f(z_2) - f(z_1)}{z_2 - z_1} = e^2 - 1, \\
f[z_2, z_3] &= f'(z_2) = 2e^2, \\
f[z_0, z_1, z_2] &= \frac{f[z_1, z_2] - f[z_0, z_1]}{z_2 - z_0} = e^2 - 3, \\
f[z_1, z_2, z_3] &= \frac{f[z_2, z_3] - f[z_1, z_2]}{z_3 - z_1} = e^2 + 1, \\
f[z_0, z_1, z_2, z_3] &= \frac{f[z_1, z_2, z_3] - f[z_0, z_1, z_2]}{z_3 - z_0} = 4, \\
p(x) &= f(z_0) + f[z_0, z_1](x - z_0) + f[z_0, z_1, z_2](x - z_0)(x - z_1) \\
&\quad + f[z_0, z_1, z_2, z_3](x - z_0)(x - z_1)(x - z_2) \\
&= 1 + 2x + (e^2 - 3)x^2 + 4x^2(x - 1).
\end{aligned}$$

(b) $f(0.5) \approx 2.597264024732662$.

□

4. (3 points) Implement an algorithm that constructs the Lagrange form of the interpolating polynomial of a function from given data points $(x_i, f(x_i))$. Use the following data and your implementation to estimate $f(0.4)$:

$$f(0.1) = -7,\ f(0.2) = -5,\ f(0.3) = -2,\ f(0.5) = 1,\ f(0.6) = 3,\ f(0.7) = 9.$$

Solution. The implementation is in `implementation.py`. Using the implementation in `problem_4.py`, $f(0.4) \approx -0.05$. □

5. (3 points) A natural cubic spline S on $[0, 2]$ is defined by

$$S(x) = \begin{cases} 1 + 2x - x^3 & 0 \le x \le 1 \\ 2 + b(x-1) + c(x-1)^2 + d(x-1)^3 & 1 \le x \le 2 \end{cases}.$$

Find the coefficients b, c, and d.

Solution. Call the first piece S_0 and the second piece S_1. Then the relevant constraints are

$$\begin{aligned} S_0'(1) &= S_1'(1), \\ S_0''(1) &= S_1''(1), \\ S_1''(2) &= 0. \end{aligned}$$

Inserting the formulas for S_0 and S_1 gives the following system for the coefficients:

$$\begin{aligned} b &= -1, \\ 2c &= -6, \\ 2c + 6d &= 0. \end{aligned}$$

Therefore, $b = -1$, $c = -3$, $d = 1$. □

6. (3 points) Consider the differentiation formula

$$f'(x) \approx \frac{1}{5h}\Big(f(x+3h) - f(x-2h) \Big).$$

Assuming that f is analytic, find the error of this approximation in terms of h. That is, find the order $r > 0$ such that the error decays like $\mathcal{O}(h^r)$.

Solution. We do Taylor expansions up to the second derivative and hide all the higher order terms: for all $x \in \mathbb{R}$, for $h > 0$ small enough, we have that

$$\begin{aligned} f(x+3h) &= f(x) + 3hf'(x) + \frac{9h^2}{2} f''(x) + \mathcal{O}(h^3), \\ f(x-2h) &= f(x) - 2hf'(x) + 2h^2 f''(x) + \mathcal{O}(h^3). \end{aligned}$$

Therefore,

$$\frac{f(x+3h) - f(x-2h)}{5h} = f'(x) + \frac{h}{2} f''(x) + \mathcal{O}(h^2),$$

so the error is $\mathcal{O}(h)$. □

7. (3 points) Use the following difference formulas to approximate the derivative of $f(x) = e^{2x}$ at $x_0 = 1$ using $h = 0.1$ and then $h = 0.05$:

$$\begin{aligned} f'(x_0) &\approx \frac{f(x_0+h) - f(x_0-h)}{2h} \quad \text{(centered difference)}, \\ f'(x_0) &\approx \frac{f(x_0+h) - f(x_0)}{h} \quad \text{(forward difference)}. \end{aligned}$$

Give your answers as decimal numbers.

Solution. We implement the difference formulas in `implementation.py` and use them in `problem_7.py` to compute:

```
true value: f'(x) = 14.7781121978613

forward difference
h = 0.1 f'(x) ~= 16.359574005034716
h = 0.05 f'(x) ~= 15.542276272740025

centered difference
h = 0.1 f'(x) ~= 14.876830175105876
h = 0.05 f'(x) ~= 14.802754702883831
```

□