MATH 437 Homework 3 (20 points)

1. The composite right rectangular rule with uniform spacing is

$$\int_a^b f(x)\,\mathrm{d}x \approx h \sum_{i=1}^{n} f(x_i).$$

   (a) (1 point) Assuming f is an analytic function, derive this formula.

   (b) (1 point) Write a computer program that implements this composite rule to integrate $f(x) = x^2 e^{-x^2}$ on $[0, 1]$ with $h = 0.1$ and $h = 0.05$.

   Hint. (a) Start with a Taylor expansion on each subinterval $[x_{i-1}, x_i]$.

   (b) Report your answer in the following format

h	integral
0.1	
0.05	

□

2. (3 points) Write a compute program that evaluates

$$\int_0^2 e^{2x} \sin(3x)\,dx$$

   using the composite trapezoidal rule with $h = 0.1$ and $h = 0.05$.

   Hint. Same as problem 1b. Recall that the composite trapezoid rule is

$$\int_a^b f(x)\,dx \approx \frac{h}{2}\left(f(a) + 2\sum_{i=1}^{n-1} f(a + ih) + f(b)\right)$$

   where $h = (b - a)/n$. □

3. (3 points) Approximate

$$\int_0^1 x^2 e^{-x}\,dx$$

   using Gaussian quadrature with $n = 2$.

Hint. You can find the Gaussian weights and points on the interval $[-1,1]$ online. Denote them by $\widehat{x}_q$, $\widehat{w}_q$. You can map them to the corresponding points x_q and weights w_q on and interval $[a,b]$ via

$$\begin{cases} x_q = \dfrac{a+b}{2} + \dfrac{b-a}{2}\widehat{x}_q, \\ w_q = \dfrac{b-a}{2}\widehat{w}_q. \end{cases}$$

□

4. (3 points) Find a, b, c, d such that the quadrature formula

$$\int_{-1}^{1} f(x)\,dx \approx af(-1) + bf(1) + cf'(-1) + df'(1)$$

has degree of precision 3.

Hint. Observe that if the rule is exact for $f(x) = 1$, $f(x) = x$, $f(x) = x^2$, and $f(x) = x^3$, then the quadrature rule has degree of precision 3. Use this to set up a system of linear equations for the coefficients. □

5. (a) (1 point) Write the explicit and implicit Euler schemes for

$$\frac{dy}{dt} = -100y, \; y(0) = 1.$$

(b) (2 points) The solution is $y(t) = e^{-100t}$. Use $h = 0.01$ and $h = 0.001$ for 3 timesteps to compute the solution $y(0.03)$ and $y(0.003)$ and the error respectively using both methods. Which method is a better approximation?

Hint. For an ODE $y'(t) = f(t, y(t))$, the explicit Euler method is

$$y_{n+1} = y_n + hf(t_n, y_n)$$

and the implicit Euler method is

$$y_{n+1} = y_n + hf(t_{n+1}, y_{n+1}).$$

For this problem, $f(t,y) = -100y$. Report your answer as follows:

h = 0.01
exact: `exact_value`
explicit Euler: `value (error)`
implicit Euler: `value (error)`

h = 0.001
exact: `exact_value`
explicit Euler: `value (error)`
implicit Euler: `value (error)`

with a short sentence or two explaining which of the two methods is better. $\square$

6. (3 points) Consider the following ODE problem:

$$\frac{dy}{dt} = -e^y, \ y(0) = 0.$$

Write a computer program that solves this problem using the implicit Euler method for $0 \le t \le 2$. For the nonlinear solve, use Newton's method. Report the error at the final time

$$e_N := |y(t_N) - y_{h,N}|,$$

where $y(t_N)$ is the exact solution at the final time time $t_N = 2$ and $y_{h,N}$ is the discrete solution at time t_N. Do all of this for timesteps (a) $h = 0.2$ and (b) $h = 0.1$. The exact solution is $y(t) = -\log(t + 1)$.

Hint. The implicit Euler scheme for this problem is

$$y_{n+1} = y_n - he^{y_{n+1}}, \ y_0 = 0.$$

Thus, at each timestep, we must solve

$$F(y) = 0$$

for a root y_{n+1}, where $F(y) := y + he^y - y_n$. Thus, Newton's method at each timestep is

$$y_{n,m+1} = y_{n,m} - \frac{F(y_{n,m})}{F'(y_{n,m})}, \ y_{n,0} = y_n.$$

We stop the Newton iterations when the residual $|F(y_{n,m+1})| < \texttt{tol}$ or $m > \texttt{max\_iter}$, and we set y_{n+1} to be the last Newton iteration at time t_n. $\square$

7. (a) (1 point) Write a computer program for solving

$$\frac{dy}{dt} = \frac{1+t}{1+y}, \ y(1) = 2, \ h = 0.02, \ 1 \le t \le 2$$

using the explicit Euler scheme.

(b) (1 point) Write a computer program to solve the same problem from the previous part using a fourth-order explicit Runge-Kutta scheme.

(c) (1 point) Compare your solutions from the previous parts to the exact solution $y(t) = \sqrt{t^2 + 2t + 6} - 1$. For both approximations, report the error at the final time $t_N = 2$:

$$e_N := |y(t_N) - y_{h,N}|.$$

Hint. Let $f(t, y) = (1 + t)/(1 + y)$. Then, the explicit Euler method for this problem is

$$y_{n+1} = y_n + hf(t_n, y_n).$$

The fourth-order explicit Runge-Kutta scheme is

$$y_{n+1} = y_n + \frac{h}{6}\Big(k_1 + 2k_2 + 2k_3 + k_4 \Big),$$

where

$$\begin{cases} k_1 = f(t_n, y_n), \\ k_2 = f(t_n + h/2, y_n + k_1 h/2), \\ k_3 = f(t_n + h/2, y_n + k_2 h/2), \\ k_4 = f(t_n + h, y_n + k_3 h). \end{cases}$$

$\square$