# MATH 437 Homework 3 (20 points)

1. The composite right rectangular rule with uniform spacing is

$$\int_a^b f(x)\,\mathrm{d}x \approx h\sum_{i=1}^{n} f(x_i).$$

   (a) (1 point) Assuming $f$ is an analytic function, derive this formula.

   (b) (1 point) Write a computer program that implements this composite rule to integrate $f(x) = x^2 e^{-x^2}$ on $[0,1]$ with $h = 0.1$ and $h = 0.05$.

*Solution.* (a) Since $f$ is analytic, on each subinterval $[x_{i-1}, x_i]$ of length $h$:

$$f(x) = f(x_i) + f'(x_i)(x - x_i) + \mathcal{O}(h^2).$$

Integrating over $[x_{i-1}, x_i]$:

$$\int_{x_{i-1}}^{x_i} f(x)\,dx = hf(x_i) + \mathcal{O}(h^2).$$

Summing over $i$ and accounting for the fact that $n \sim 1/h$:

$$\int_a^b f(x)\,dx = h\sum_{i=1}^{n} f(x_i) + \mathcal{O}(h).$$

(b) I made a typo when copying the question from the professor. His question says to integrate over $[0,2]$, while I said to integrate over $[0,1]$. Either answer is fine. here are the results for $[0,2]$:

| $h$ | integral |
|---|---|
| h = 0.1 | integral = 0.42620506905911787 |
| h = 0.05 | integral = 0.4245108338353924 |

and here are the results for $[0,1]$:

| $h$ | integral |
|---|---|
| h = 0.1 | integral = 0.20786611213159514 |
| h = 0.05 | integral = 0.1986693190548683 |

$\square$

2. (3 points) Write a compute program that evaluates

$$\int_0^2 e^{2x} \sin(3x)\, dx$$

using the composite trapezoidal rule with $h = 0.1$ and $h = 0.05$.

*Solution.*

| $h$ | integral |
|---|---|
| 0.1 | -14.111007098611367 |
| 0.05 | -14.18820404213729 |

□

3. (3 points) Approximate

$$\int_0^1 x^2 e^{-x}\, dx$$

using Gaussian quadrature with $n = 2$.

*Solution.* For Gauss quadrature, $n$ usually refers to the number of points. I was ambiguous with what $n$ meant here, since it can also be used to refer to the last index when counting the points as $x_0$, $x_1$, ..., $x_n$. In the latter case, there are $n+1$ quadrature points. Therefore, some people did 2 point quadrature (what was expected), and some people did 3 point quadrature with points $x_0$, $x_1$, $x_2$. I accept either answer.

For 2 point Gauss quadrature, the integral is approximately 0.15941043096637894. For 3 points, the integral is approximately 0.16059538680891924 □

4. (3 points) Find $a, b, c, d$ such that the quadrature formula

$$\int_{-1}^1 f(x)\, dx \approx af(-1) + bf(1) + cf'(-1) + df'(1)$$

has degree of precision 3.

*Solution.* We set up the following system of equations:

$$\begin{aligned}
2 &= \int_{-1}^1 1\, dx = a + b + c + d, \\
0 &= \int_{-1}^1 x\, dx = -a + b + c + d, \\
2/3 &= \int_{-1}^1 x^2\, dx = a + b - 2c + 2d, \\
0 &= \int_{-1}^1 x^3\, dx = -a + b + 3c + 3d.
\end{aligned}$$

Solving it yields $a = 1$, $b = 1$, $c = 1/3$, and $d = -1/3$. □

5. (a) (1 point) Write the explicit and implicit Euler schemes for

$$\frac{dy}{dt} = -100y, \ y(0) = 1.$$

(b) (2 points) The solution is $y(t) = e^{-100t}$. Use $h = 0.01$ and $h = 0.001$ for 3 timesteps to compute the solution $y(0.03)$ and $y(0.003)$ and the error respectively using both methods. Which method is a better approximation?

*Solution.* (a) Explicit Euler:

$$y^{n+1} = y^n - 100hy^n.$$

Implicit Euler:

$$y^{n+1} = y^n - 100hy^{n+1}.$$

Values and errors:

h = 0.01
exact: 0.049787068367863944
explicit Euler (error): 0.0 (0.049787068367863944)
implicit Euler (error): 0.125 (0.07521293163213605)

h = 0.001
exact: 0.7408182206817179
explicit Euler (error): 0.7290000000000001 (0.011818220681717784)
implicit Euler (error): 0.7513148009015777 (0.010496580219859775)

While the explicit Euler method gives a lower error for $h = 0.01$, it has the wrong qualitative behavior, since $e^{-100t} > 0$ for all $t$ but the method produces $y = 0$ for all $t$. The implicit Euler method always has the correct qualitative behavior for all $h > 0$ and gives a better approximation than the explicit Euler method for $h = 0.001$. Therefore, for this test case, we prefer the implicit Euler method.

□

6. (3 points) Consider the following ODE problem:

$$\frac{dy}{dt} = -e^y, \ y(0) = 0.$$

Write a computer program that solves this problem using the implicit Euler method for $0 \leq t \leq 2$. For the nonlinear solve, use Newton's method. Report the error at the final time

$$e_N := |y(t_N) - y_{h,N}|,$$

where $y(t_N)$ is the exact solution at the final time time $t_N = 2$ and $y_{h,N}$ is the discrete solution at time $t_N$. Do all of this for timesteps (a) $h = 0.2$ and (b) $h = 0.1$. The exact solution is $y(t) = -\log(t + 1)$.

*Solution.*

| $h$ | error |
|---|---|
| 0.2 | 0.034764531045191394 |
| 0.1 | 0.01786509930941249 |

□

7. (a) (1 point) Write a computer program for solving

$$\frac{dy}{dt} = \frac{1+t}{1+y},\ y(1) = 2,\ h = 0.02,\ 1 \leq t \leq 2$$

using the explicit Euler scheme.

(b) (1 point) Write a computer program to solve the same problem from the previous part using a fourth-order explicit Runge-Kutta scheme.

(c) (1 point) Compare your solutions from the previous parts to the exact solution $y(t) = \sqrt{t^2 + 2t + 6} - 1$. For both approximations, report the error at the final time $t_N = 2$:

$$e_N := |y(t_N) - y_{h,N}|.$$

*Solution.*

| method | error |
|---|---|
| explicit Euler | 0.001203385963981951 |
| RK4 | 3.7623237858497305e-12 |

□