MATH 437 Homework 4 (20 points)

1. (5 points) Use the Adams-Bashforth four-step method to solve

$$\frac{dy}{dt} = te^{3t} - 2y$$

on the interval $0 \le t \le 1$ with initial condition $y(0) = 0$. Use step size $h = 0.2$ and starting values based on the exact solution

$$y(t) = \frac{t}{5}e^{3t} - \frac{1}{25}e^{3t} + \frac{1}{25}e^{-2t}.$$

Output the discrete solution y_n at each timestep t_n.

Hint. See `problem_1.py` in the `code-templates`. □

2. (5 points) Derive the Adams–Moulton two-step implicit method by using the appropriate form of an interpolating polynomial.

Hint. Multi-step methods to solve problems of the form

$$y'(t) = f(t, y(t))$$

start by integrating both sides from t_n to t_{n+1}:

$$y(t_{n+1}) - y(t_n) = \int_{t_n}^{t_{n+1}} f(t, y(t))\, dt.$$

Now, the idea is to replace the function $\phi(t) := f(t, y(t))$ by a Lagrange polynomial based on the discrete time points t_n. For the Adams–Moulton 2-step method, the points used are t_{n-1}, t_n, and t_{n+1}. To simplify the computations, assume that $t_n - t_{n-1} = t_{n+1} - t_n = h$. Let L_{n-1}, L_n, and L_{n+1} be the Lagrange polynomials interpolating $\phi(t)$ at the points t_{n-1}, t_n, and t_{n+1}. To compute the integrals of these polynomials, it's useful to make a change of variables $t \mapsto s = (t - t_{n-1})/(t_n - t_{n-1})$ and then integrate with respect to s from 0 to 1. □

3. (5 points) Investigate the stability of the difference method

$$w_{i+1} = w_i + hf(t_i, w_i) + h^2 f(t_{i-1}, w_{i-1}).$$

Assume that f satisfies a Lipschitz condition on $\{(t, w) \mid a \leq t \leq b \text{ and } -\infty < w < \infty\}$ in the variable w with constant $L > 0$, and assume that f is continuous in t. See section 5.10 in the textbook. In particular, see the discussion on multi-step methods. Also, see Lecture 10 in the class notes.

Hint. For stability of this multi-step method, rewrite the equation in the form

$$w_{i+1} = w_i + hF(h, t_i, w_i, w_{i-1}).$$

Then, to investigate the stability of this method, we verify the following:

(a) If $f = 0$, then $F = 0$.

(b) F satisfies a Lipschitz condition in w_i and w_{i-1}: there is a constant $C > 0$ such that

$$|F(h, t_i, w_i, w_{i-1}) - F(h, t_i, z_i, z_{i-1})| \leq C(|w_i - z_i| + |w_{i-1} - z_{i-1}|).$$

(c) Determine the characteristic polynomial $p(\lambda)$ of the method, and check its roots. See Definitions 5.22 and 5.23 in the book.

If the first 2 conditions are not satisfied, the method is unstable. If the first 2 conditions are satisfied, then check Definitions 5.22 and 5.23 to determine if the method is unstable, weakly stable, or strongly stable. □

4. (5 points) Solve the following stiff differential equation

$$y' = -20y + 21e^t$$

on the interval $0 \leq t \leq 1$ with initial condition $y(0) = 2$. Use the explicit Euler and implicit Euler methods step sizes $h = 0.1$ and $h = 0.01$. Turn in plots of the solutions and the exact solution $y(t) = e^t + e^{-20t}$.

Hint. See `problem_4.py` in the `code-templates`. □