MATH 437 Homework 4 (20 points)

1. (5 points) Use the Adams-Bashforth four-step method to solve

$$\frac{dy}{dt} = te^{3t} - 2y$$

on the interval $0 \leq t \leq 1$ with initial condition $y(0) = 0$. Use step size $h = 0.2$ and starting values based on the exact solution

$$y(t) = \frac{t}{5}e^{3t} - \frac{1}{25}e^{3t} + \frac{1}{25}e^{-2t}.$$

Output the discrete solution y_n at each timestep t_n.

Solution.

t_n	y_n
0.000000e+00	0.000000e+00
2.000000e-01	2.681280e-02
4.000000e-01	1.507778e-01
6.000000e-01	4.960196e-01
8.000000e-01	1.296126e+00
1.000000e+00	3.146140e+00

□

2. (5 points) Derive the Adams–Moulton two-step implicit method by using the appropriate form of an interpolating polynomial.

Solution. The Lagrange interpolating polynomials at the points t_{n-1}, t_n, t_{n+1} are

$$\begin{cases} L_{n-1}(t) = \dfrac{(t-t_i)(t-t_{i+1})}{2h^2}, \\ L_n(t) = \dfrac{(t-t_{i-1})(t-t_{i+1})}{-h^2}, \\ L_{n+1}(t) = \dfrac{(t-t_{i-1})(t-t_i)}{2h^2}. \end{cases}$$

We integrate each one from $t = t_i$ to $t = t_{i+1}$. It is convenient to switch variables $t = t_i + sh$, $dt = h\,ds$. Then,

$$\int_{t_i}^{t_{i+1}} L_{i-1}(t)\,dt = \frac{h}{2}\int_0^1 s(s-1)\,ds = -\frac{h}{12},$$
$$\int_{t_i}^{t_{i+1}} L_i(t)\,dt = h\int_0^1 (1+s)(1-s)\,ds = \frac{2h}{3},$$
$$\int_{t_i}^{t_{i+1}} L_{i+1}(t)\,dt = \frac{h}{2}\int_0^1 (1+s)s\,ds = \frac{5h}{12}.$$

Therefore, the method is

$$y_{n+1} = y_n - \frac{h}{12}f(t_{n-1}, y_{n-1}) + \frac{2h}{3}f(t_n, y_n) + \frac{5h}{12}f(t_{n+1}, y_{n+1}).$$

□

3. (5 points) Investigate the stability of the difference method

$$w_{i+1} = w_i + hf(t_i, w_i) + h^2 f(t_{i-1}, w_{i-1}).$$

Assume that f satisfies a Lipschitz condition on $\{(t,w) \mid a \le t \le b \text{ and } -\infty < w < \infty\}$ in the variable w with constant $L > 0$, and assume that f is continuous in t. See section 5.10 in the textbook. In particular, see the discussion on multi-step methods. Also, see Lecture 10 in the class notes.

Solution. Set $F(h, t_i, w_i, w_{i-1}) = f(t_i, w_i) + hf(t_i - h, w_{i-1})$. Then, if $f = 0$, $F = 0$. Next, for $h, t_i, w_i, w_{i-1}, z_i, z_{i-1}$,

$$\begin{aligned} &|F(h, t_i, w_i, w_{i-1}) - F(h, t_i, z_i, z_{i-1})| \\ &\qquad \le |f(t_i, w_i) - f(t_i, z_i)| + h|f(t_i - h, w_{i-1}) - f(t_i - h, z_{i-1})| \\ &\qquad\qquad \le L|w_i - z_i| + hL|w_{i-1} - z_{i-1}|. \end{aligned}$$

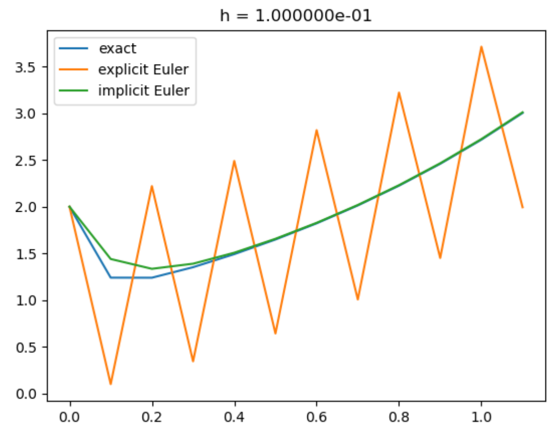
Without loss of generality, assume $h < 1$. Then, F satisfies a Lipschitz condition with constant $L > 0$. The characteristic polynomial is $p(\lambda) = \lambda - 1$. It has a single root $\lambda = 1$, so it is strongly stable. □

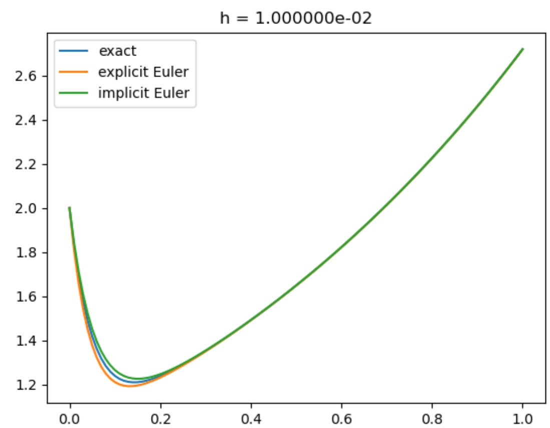
4. (5 points) Solve the following stiff differential equation

$$y' = -20y + 21e^t$$

on the interval $0 \leq t \leq 1$ with initial condition $y(0) = 2$. Use the explicit Euler and implicit Euler methods step sizes $h = 0.1$ and $h = 0.01$. Turn in plots of the solutions and the exact solution $y(t) = e^t + e^{-20t}$.

Solution.





□