MATH 437 Homework 6 (20 points)

1. (2 points) Find the linear least-squares polynomial approximation on $[-1, 1]$ to

$$f(x) = x^3.$$

Hint. Let $p(a, b; x) = a + bx$, where a and b will be determined so that p is the linear least-squares approximation to f. The least-squares approximation minimizes the error

$$e(a, b) := \int_{-1}^{1} (f(x) - p(a, b; x))^2 \, dx.$$

Thus, the partial derivatives of e with respect to a and b should be 0:

$$\partial_a e = 0,$$
$$\partial_b e = 0.$$

Alternatively, requiring the partial derivatives vanish is equivalent to requiring the difference

$$g(a, b; x) := f(x) - p(a, b; x)$$

is orthogonal to the linear basis monomials 1 and x:

$$\partial_a e = 0 \iff \int_{-1}^{1} g(a, b; x) \, dx = 0,$$
$$\partial_b e = 0 \iff \int_{-1}^{1} g(a, b; x) x \, dx = 0.$$

□

2. (3 points) Use the Gram–Schmidt process to construct an orthogonal basis $\phi_0(x)$, $\phi_1(x)$, $\phi_2(x)$ for the space of polynomials of degree 2 on the interval $[0, 2]$.

Hint. For any functions f, g on $[0, 2]$, we let

$$(f, g) := \int_{0}^{2} f(x) g(x) \, dx.$$

We start with the non-orthogonal monomial basis $\psi_0(x) = 1$, $\psi_1(x) = x$, $\psi_2(x) = x^2$, and we orthogonalize via Gram–Schmidt. We set $\phi_0 := \psi_0$. Then, we project ψ_1 onto $\operatorname{span} \phi_0$:

$$p_1 := \frac{(\psi_1, \phi_0)}{(\phi_0, \phi_0)} \phi_0.$$

By removing p_1 from ψ_1, we obtain ϕ_1:

$$\phi_1 := \psi_1 - p_1.$$

To construct ϕ_2, we project ψ_2 onto $\operatorname{span}\{\phi_0, \phi_1\}$:

$$p_2 := \frac{(\psi_2, \phi_0)}{(\phi_0, \phi_0)} \phi_0 + \frac{(\psi_2, \phi_1)}{(\phi_1, \phi_1)} \phi_1.$$

By removing this from ψ_2, we obtain ϕ_2:

$$\phi_2 := \psi_2 - p_2.$$

□

3. (3 points) Compute the first three iterations of the power method without normalization (see notes) with the following matrix:

$$A := \begin{bmatrix} 4 & 2 & 1 \\ 2 & 4 & 0 \\ 1 & 0 & 4 \end{bmatrix}.$$

Start with initial eigenvector $\boldsymbol{x}_0 = (1,1,1)^T$. Report the approximate eigenvalue and eigenvector $(\mu_i, \boldsymbol{x}_i)$ for $i = 1,2,3$.

Hint. We follow the standard power method without normalization:

$$x_{k+1} = Ax_k, \qquad \mu_{k+1} := \frac{x_{k+1,p}}{x_{k,p}},$$

where p is the smallest index that all $x_{k,p}$ are (eventually) nonzero. We iterate for $k = 0,1,2$. □

4. (3 points) Let A be an $n \times n$ matrix with n distinct eigenvalues ordered such that

$$|\lambda_1| > |\lambda_2| > \cdots > |\lambda_n|$$

and with corresponding eigenvectors $v_1, \ldots, v_n$. Let $\beta_2, \ldots, \beta_n \in \mathbb{R}$, where $\beta_2 \neq 0$, and set

$$x_0 := \sum_{i=2}^{n} \beta_i v_i,$$

so that x_0 is in the span of all but the first eigenvectors. Prove that the sequence of approximate eigenvalues μ_k obtained from the power method starting from this initial condition converges to the second-largest eigenvalue λ_2.

Hint. Show the following steps.

(a) The power method (without normalization) to construct the approximating sequence of eigenvectors x_k is

$$x_k := Ax_{k-1}.$$

Show that

$$x_k = \beta_2 \lambda_2^k \left(v_2 + \sum_{i=3}^{n} \frac{\beta_i}{\beta_2} \left(\frac{\lambda_i}{\lambda_2} \right)^k v_i \right).$$

(b) Let

$$r_k := v_2 + \sum_{i=3}^{n} \frac{\beta_i}{\beta_2} \left(\frac{\lambda_i}{\lambda_2} \right)^k v_i.$$

Show that $r_k \to v_2$ as $k \to \infty$.

(c) Show that there is an index p such that, for k large enough, all $r_{k,p} \neq 0$ and all $x_{k,p} \neq 0$, where $r_{k,p}$ is the pth component of r_k, and similarly for $x_{k,p}$.

(d) Show that, for some index p and for k large enough, the ratios $x_{k+1,p}/x_{k,p}$ and $r_{k+1,p}/r_{k,p}$ are well-defined, and

$$\frac{x_{k+1,p}}{x_{k,p}} = \lambda_2 \frac{r_{k+1,p}}{r_{k,p}} \to \lambda_2$$

as $k \to \infty$.

(e) The approximate eigenvalues for the power iteration are defined as

$$\mu_{k+1} := \frac{x_{k+1,p}}{x_{k,p}},$$

where p is from part c. Conclude that $\mu_k \to \lambda_2$.

□

5. (3 points) Use the Gershgorin Circle Theorem to find bounds on the eigenvalues for

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}.$$

Hint. See Theorem 9.1 in the textbook. The eigenvalues lie within the union of the *Gershgorin circles* in the complex plane:

$$R_i := \{z \in \mathbb{C} \mid |z - A_{i,i}| \leq \sum_{j \neq i} |A_{i,j}|\}.$$

This will tell you that all eigenvalues satisfy an inequality of the form $|\lambda - z_0| \leq r_0$ for some real numbers z_0, r_0. Find z_0 and r_0. Since the matrix is real and symmetric, the eigenvalues are actually all real-valued, so you can deduce from the previous inequality that all eigenvalues lie in some interval $[a, b]$. Find a and b. □

6. For the following set of vectors:

(a) (1 point) Show that the set is linearly independent.

(b) (2 points) Use the Gram–Schmidt process to orthogonalize the set.

$$v_1 = (1, 1, 1, 1)^T, \qquad v_2 = (0, 2, 2, 2)^T, \qquad v_3 = (1, 0, 0, 1)^T.$$

Hint. (a) Suppose $c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$. Then, by looking at components, set up a linear system for the c_i and conclude that the $c_i = 0$.

(b) We let $(v, w) := \sum_i v_i w_i$ denote the standard dot product of vectors. We let ϕ_1, ϕ_2, ϕ_3 denote the desired orthogonal vectors. We set $\phi_1 = v_1$. Then, we project v_2 onto $\operatorname{span} \phi_1$:

$$p_2 := \frac{(v_2, \phi_1)}{(\phi_1, \phi_1)} \phi_1.$$

Then, we remove this from v_2 to get ϕ_2:

$$\phi_2 = v_2 - p_2.$$

Now, we project v_3 onto $\operatorname{span}\{\phi_1, \phi_2\}$:

$$p_3 = \frac{(v_3, \phi_1)}{(\phi_1, \phi_1)} \phi_1 + \frac{(v_3, \phi_2)}{(\phi_2, \phi_2)} \phi_2.$$

Then, we subtract from v_3 to obtain ϕ_3:

$$\phi_3 = v_3 - p_3.$$

□

7. (3 points) Construct an orthogonal matrix Q such that $Q^T A Q = D$, where D is a diagonal matrix and

$$A = \begin{bmatrix} 4 & 2 & 1 \\ 2 & 4 & 0 \\ 1 & 0 & 4 \end{bmatrix}$$

Hint. Since A is symmetric, the orthogonal matrix Q that we seek has columns consisting of normalized eigenvectors of A, and the diagonal matrix has entries consisting of the eigenvalues of A (see Corollary 9.17 in the textbook). We first find the eigenvalues by solving the polynomial equation

$$\det(A - \lambda I) = 0.$$

The diagonal matrix D is

$$D = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{bmatrix},$$

where λ_i are the distinct eigenvalues solved for from the previous equation. Let $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, $\boldsymbol{x}_3$ be the corresponding normalized eigenvectors, i.e. $A\boldsymbol{x}_i = \lambda_i \boldsymbol{x}_i$, and $\|\boldsymbol{x}_i\|_{\ell_2}^2 = \sum_j x_{ij}^2 = 1$. Using these equations will let you uniquely determine the eigenvectors. The matrix Q is formed by setting the columns to the normalized eigenvectors:

$$Q = \begin{bmatrix} \boldsymbol{x}_1 & \boldsymbol{x}_2 & \boldsymbol{x}_3 \end{bmatrix}.$$

□