MATH 437 Homework 8 (20 points)

1. (5 points) Consider the following parabolic PDE:

$$\partial_t u - 0.5\partial_x^2 u = x^2$$

for $t > 0$ and $x \in (0, 1)$ with initial condition $u(x, 0) = 0$ and boundary conditions $u(0, t) = u(1, t) = 0$. Solve this PDE using forward Euler in time and centered differences in space. Let $h = 0.1$ denote the spatial mesh size and $\tau = 0.01$ denote the timestep size. Solve until final time $T = 0.1$ and plot the solution at the final time.

Hint. Let $t_n := n\tau$ and $x_i := ih$ denote the discrete time and space points. Let u_i^n denote the discrete approximation to $u(x_n, t_i)$. Recall that forward Euler for a function of a single variable $u(t)$ reads

$$\frac{u^{n+1} - u^n}{\tau}$$

to approximate $u'(t)$ at $t = t_n$. Also recall that the second-order centered difference scheme for a function of a single variable $u(x)$ reads

$$\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2}$$

to approximate $u''(x)$ at $x = x_i$. Use these and the PDE to derive a difference equation involving the u_i^n. Don't forget to apply the initial condition at $t_0 = 0$ and the boundary conditions at $x_0 = 0$ and $x_M = 1$. See `problem_1.py` for a code template. □

2. (5 points) Consider the following boundary-value problem

$$-y'' + y = x$$

with $y(0) = y(1) = 0$. Solve this problem using the finite element method with mesh size $h = 1/3$. Report the values (x_j, y_j) of the solution, where $x_j = ih$ and y_j is the approximation to $y(x_j)$.

Hint. We let $\{\phi_j\}_{j=1,2}$ be the standard piecewise linear finite element basis using the interior grid points $x_1 = 1/3$, $x_2 = 2/3$. We expand the finite-element solution y_h in this basis:

$$y_h = y_1\phi_1 + y_2\phi_2,$$

where y_j is the coefficient with respect to ϕ_j, which happens to approximate the exact solution $y(x_j)$ due to the properties of the ϕ_j. By replacing y with y_h in the PDE, multiplying by ϕ_i, and integrating by parts, derive a 2×2 system of equations of the form

$$A\vec{y} = \vec{b},$$

where A is a matrix whose entries consist of integrals involving the ϕ_j and their derivatives, $\vec{b}$ is a vector whose entries involve integrating the right-hand side multiplied by ϕ_i, and $\vec{y}$ is the vector of coefficients y_i.

The formula for ϕ_j is

$$\phi_j(x) = \begin{cases} \frac{x-x_{j-1}}{h}, & x \in [x_{j-1}, x_j] \\ \frac{x_{j+1}-x}{h}, & x \in [x_j, x_{j+1}] \\ 0, & \text{else} \end{cases},$$

where $x_0 = 0$ and $x_3 = 1$. Therefore, by using a computer to assist the computations, assemble A and $\vec{b}$ and solve for $\vec{y}$. See `problem_2.py` for a code template. □

3. Consider the following boundary-value problem

$$y'' + 2y^3 = 1$$

with $y(0) = y(1) = 0$. Let $h = 1/3$.

(a) (3 points) Write the second-order centered finite difference method for this equation.

(b) (2 points) Write Newton's method for the resulting linear system and perform 2 iterations starting from an $\vec{y}^{(0)} = \vec{0}$.

(a) *Hint.* See the hint for question 1. Letting y_i be the approximate solution at $x_i = ih$ with $i = 0, 1, 2, 3$, the PDE gives 2 equations at x_1, x_2 while the boundary conditions give 2 equations at x_0 and x_3. □

(b) *Hint.* Let $\vec{y} = (y_1, y_2)^T$. Using the 2 difference equations from the previous part, eliminate y_0 and y_3 using the boundary conditions. From the resulting equations, define a function $\vec{F}(\vec{y}) \in \mathbb{R}^2$ such that solving the system in the previous problem is equivalent to finding $\vec{y}$ such that

$$\vec{F}(\vec{y}) = \vec{0}.$$

Newton's method then reads

$$\vec{y}^{(n+1)} = \vec{y}^{(n)} - \nabla\vec{F}(\vec{y}^{(n)})^{-1}\vec{F}(\vec{y}^{(n)}).$$

Compute $\nabla\vec{F}(\vec{y})$ by hand, and use a computer to perform the iterations, starting with $\vec{y}^{(0)} = \vec{0}$. See `problem_3.py` for a template. □

4. (a) (3 points) Use the continuation method to solve

$$\begin{aligned} x_1 + x_2^2 - 6 &= 0, \\ -x_1 + x_2 - 1 &= 0, \end{aligned}$$

with initial condition $x = (0, 0)$. Write down the differential equation for the solution.

(b) (2 points) Perform 5 iterations of the forward Euler method on the differential equation from the previous step with $\tau = 0.2$.

(a) *Hint.* Use the system of equations to define a function $\vec{F} : \mathbb{R}^2 \to \mathbb{R}^2$ such that solving the system is equivalent to solving

$$\vec{F}(\vec{x}) = \vec{0}.$$

Then, the continuation method considers the function

$$\vec{G}(\lambda, \vec{x}) := \lambda F(\vec{x}) + (1-\lambda)(\vec{F}(\vec{x}) - \vec{F}(\vec{0})).$$

We now suppose that there is a smooth function $\vec{x}(\lambda)$ such that

$$\vec{G}(\lambda, \vec{x}(\lambda)) = \vec{0}$$

for all λ. In particular, this implies that $\vec{G}(1, \vec{x}(1)) = \vec{F}(\vec{x}(1)) = \vec{0}$, so $\vec{x}(1)$ is a solution to the original problem. We also observe that $\vec{G}(0, \vec{x}(0)) = \vec{F}(\vec{x}(0)) - \vec{F}(\vec{0}) = \vec{0}$, which is trivially solved by requiring $\vec{x}(0) = \vec{0}$. Now, if we differentiate with respect to λ:

$$\begin{aligned} \vec{0} &= \frac{d}{d\lambda}\vec{G}(\lambda, \vec{x}(\lambda)) \\ &= \partial_\lambda \vec{G}(\lambda, \vec{x}(\lambda)) + \nabla_{\vec{x}}\vec{G}(\lambda, \vec{x}(\lambda))\vec{x}'(\lambda), \end{aligned}$$

where $\nabla_{\vec{x}}\vec{G}$ denotes the Jacobian matrix of $\vec{G}$ with entries

$$(\nabla_{\vec{x}}\vec{G})_{i,j} := \partial_{x_j} G_i.$$

Solving, we see that $\vec{x}(\lambda)$ satisfies the following ODE system:

$$\vec{x}'(\lambda) = -\nabla_{\vec{x}}\vec{G}(\lambda, \vec{x}(\lambda))^{-1}\partial_\lambda \vec{G}(\lambda, \vec{x}(\lambda))$$

with initial condition $\vec{x}(0) = \vec{0}$. Solving this ODE with a numerical method that gives an approximation to $\vec{x}(1)$ gives us an approximate solution to $\vec{F}(\vec{x}) = \vec{0}$. This simplifies:

$$\begin{aligned} \partial_\lambda \vec{G}(\lambda, \vec{x}) &= \vec{F}(\vec{0}), \\ \nabla_{\vec{x}}\vec{G}(\lambda, \vec{x}) &= \nabla \vec{F}(\vec{x}). \end{aligned}$$

Thus, the ODE to solve is

$$\vec{x}' = -\nabla \vec{F}(\vec{x})^{-1}\vec{F}(\vec{0})$$

with $\vec{x}(\vec{0}) = \vec{0}$. Compute $\vec{F}(\vec{0})$ and $\nabla \vec{F}(\vec{x})$ for this particular problem. □

(b) *Hint.* See the hint for problem 1 for the forward Euler method. Using a computer to assist the computations, compute 5 iterations. See `problem_4.py` for a template. □