MATH 437 Notes

Jordan Hoffart

Contents

1 Lecture 1

1.1 Bisection method

Let $f(x)$ be a continuous function on the interval $[a, b]$.

Proposition 1 (Existence of roots)**.** *If* $f(a)f(b) < 0$*, then there is a point* $p \in (a, b)$ *such that* $f(p) = 0$.

Proof. If $f(a)f(b) < 0$, then $f(a)$ and $f(b)$ have opposite signs. That is, $f(a) > 0$ and $f(b) < 0$, or $f(a) < 0$ and $f(b) > 0$. By the Intermediate Value Theorem, f attains all possible values between $f(a)$ and $f(b)$. In particular, there is a point $p \in (a, b)$ where $f(p) = 0$. □

The bisection method is an algorithm to find the point p. The algorithm is as follows for the case that $f(a) < 0 < f(b)$.

Algorithm 1 Bisection method

$x_{\text{left}} := a$, $x_{\text{right}} := b$, $n := 0$, $x := (x_{\text{left}} + x_{\text{right}})/2$
while $|x_{\text{left}} - x_{\text{right}}| >$ tol and $n \le n_{\max}$ **do**
    if $f(x) = 0$ **then**
        return x
    else if $f(x) < 0$ **then**
        $x_{\text{left}} \leftarrow x$
    else
        $x_{\text{right}} \leftarrow x$
    end if
    $n \leftarrow n + 1$
    $x \leftarrow (x_{\text{left}} + x_{\text{right}})/2$
end while
return x

1.2 Fixed point methods

Given a continuous function $g(x)$, suppose we want to solve the equation $x = g(x)$. One possible iterative method is defined by

$$x_{n+1} = g(x_n), \tag{1}$$

where we provide a starting value x_0. Whether or not this converges to a solution as $n \to \infty$ depends on the properties of g and the starting value x_0.

Theorem 2 (Existence of fixed points). *If $g(x) \in [a, b]$ for all $x \in [a, b]$, then g has a fixed point in $[a, b]$.*

Proof. Let $h(x) = g(x) - x$. Then $h(a) \le 0$, $h(b) \ge 0$ and h is continuous. If $h(a) = 0$, then a is a fixed point of g. If $h(b) = 0$, then b is a fixed point of g. If $h(a)$ and $h(b)$ are both nonzero, then $h(a) < 0 < h(b)$. By the previous proposition, there exists $x_0 \in (a, b)$ such that $h(x_0) = 0$, i.e. $g(x_0) = x_0$. □

Theorem 3 (Convergence of fixed-point methods). *Suppose g is differentiable, and there exists k such that $|g'(x)| \le k < 1$ for all $x \in [a, b]$. Then, g has a unique fixed point, and the iterative method* (1) *converges to this point for any initial value $x_0 \in [a, b]$.*

Proof. By the Mean Value Theorem, for distinct $x, y \in [a, b]$, there exists $z \in [a, b]$ such that

$$g(x) - g(y) = g'(z)(x - y).$$

Therefore, since $|g'(z)| \le k < 1$, we have that

$$|g(x) - g(y)| \le k|x - y| < |x - y|$$

for all distinct $x, y \in [a, b]$.

Now, let $x_0 \in [a, b]$, and set $x_{n+1} = g(x_n)$ for all $n \geq 0$. From above, for all $n \geq 0$,

$$|x_{n+2} - x_{n+1}| = |g(x_{n+1}) - g(x_n)| \leq k|x_{n+1} - x_n|.$$

By repeating this, we have

$$|x_{n+2} - x_{n+1}| \leq k^{n+1}|x_1 - x_0|$$

for all n. Therefore, for any $m > n \geq 0$, by writing $m = n + (m - n)$,

$$\begin{aligned} |x_m - x_n| \leq |x_{n+(m-n)} - x_{n+(m-n-1)}| \\ + |x_{n+(m-n-1)} - x_{n+(m-n-2)}| + \cdots + |x_{n+1} - x_n| \\ \leq (k^{m-n-1} + k^{m-n-2} + \cdots + k^n)|x_1 - x_0|. \end{aligned} \quad (2)$$

Since $k < 1$, the terms in the last inequality are the Cauchy tail of the convergent geometric series $\sum_i k^i$. Therefore, $|x_m - x_n| \to 0$ as $m, n \to \infty$, so the sequence $(x_n)_n$ is a Cauchy sequence of real numbers. The sequence therefore must converge to some number p.

Since $g(x_n) = x_{n+1}$ and g is continuous, taking limits of this equation yields $g(p) = p$, so p is a fixed point of g. If q is another fixed point of g, then, from above,

$$|p - q| = |g(p) - g(q)| < |p - q|,$$

which is a contradiction, so p is the only fixed point of g. □

2 Lecture 2

2.1 Newton's method

Newton's method is a fixed-point method to find the roots of a differentiable function $f(x)$. It is defined by the following algorithm:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \quad (3)$$

If we set $g(x) = f(x)/f'(x)$, then the above equation is of the form (1), so that Newton's method is indeed a fixed-point method.

Lemma 4. *Suppose f is twice differentiable and $f'(p) \neq 0$. Let $g(x) = x - f(x)/f'(x)$.*

1. *p is a fixed point of g iff $f(p) = 0$.*
2. *If $f(p) = 0$, then $g'(p) = 0$.*

Proof. 1. If p is a fixed point of g, then $p = g(p) = p - f(p)/f'(p)$, so $f(p) = 0$. Conversely, if $f(p) = 0$, then $g(p) = p - f(p)/f'(p) = p$.

2.
$$g'(x) = 1 - \frac{f'(x)^2 - f(x)f''(x)}{f'(x)^2} = \frac{f(x)f''(x)}{f'(x)^2}.$$

Since $f(p) = 0$, $g'(p) = 0$.

□

Theorem 5 (Convergence of Newton's method)**.** *Suppose f is twice differentiable, has a root at p and $f'(p) \neq 0$. For an initial value x_0 sufficiently close to p, Newton's method converges to p.*

Proof. We let $g(x) = f(x)/f'(x)$. Then, g is continuously differentiable, and, by the previous lemma, p is a fixed point of g and $g'(p) = 0$. Therefore, there exists $\delta > 0$ such that, whenever $|x - p| \leq \delta$, $|g'(x)| \leq 1/2 < 1$. By using Theorem 3 with $k = 1/2$, $a = p - \delta$, $b = p + \delta$, we conclude that whenever $x_0 \in [a, b]$, Newton's method converges to p. □

2.2 Quadratic convergence of Newton's method

Definition 6 (Order of convergence)**.** Suppose that a sequence $x_n \to p$ as $n \to \infty$. We say that the sequence converges with order $r > 0$ if there is a constant $0 \leq \lambda < \infty$ such that

$$|x_{n+1} - p| \leq \lambda |x_n - p|^r \tag{4}$$

for all n sufficiently large. For $r = 1$, we say the sequence converges linearly, and for $r = 2$, we say the sequence converges quadratically.

Theorem 7 (Quadratic convergence of Newton's method)**.** *Let f be a 3-times continuously differentiable function with a root at p and $f'(p) \neq 0$. Suppose that an initial value x_0 is chosen sufficiently close to p so that Newton's method converges to p. Then, the method converges quadratically.*

Proof. We let $g(x) = f(x)/f'(x)$. Then, g is a twice continuously differentiable function. Using Taylor's Theorem around p, for all n, there exists ξ_n between x_n and p such that

$$x_{n+1} = g(x_n) = g(p) + g'(p)(x_n - p) + g''(\xi_n)(x_n - p)^2.$$

From Lemma 4, $g(p) = p$ and $g'(p) = 0$, so

$$|x_{n+1} - p| = |g''(\xi_n)||x_n - p|^2.$$

There exists $N > 0$ such that $|x_n - p| \leq 1$ for all $n \geq N$. Thus, for all $n \geq N$, ξ_n lies in the interval $[p - 1, p + 1]$. Since g'' is continuous on the closed and bounded interval $[p - 1, p + 1]$, we may set

$$\lambda := \max_{\xi \in [p-1, p+1]} |g''(\xi)|.$$

Then, we conclude that

$$|x_{n+1} - p| \leq \lambda |x_n - p|^2$$

when $n \geq N$, so Newton's method converges quadratically. □

2.3 Secant method

In Newton's method, we may replace $f'(x)$ by a backward difference approximation

$$f'(x_n) \approx \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}.$$

Doing so gives us the secant method:

$$x_{n+1} = x_n - \frac{f(x_n)}{f(x_n) - f(x_{n-1})}(x_n - x_{n-1}), \tag{5}$$

where now we must provide 2 initial conditions x_0, x_1.