2D Linear FEM Notes

$$\int_\Omega \left(-\Delta u + qu = f\right) v \qquad \int_\Omega \nabla\cdot(\nabla u\, v) = \int_\Omega \nabla u\cdot\nabla v + \int_\Omega \Delta u\, v$$

$$\int_{\partial\Omega} \nabla u\, v\cdot n \underset{\substack{\uparrow\\ \text{BC}}}{=} \int_{\partial\Omega} gv$$

$$\longrightarrow \quad \int_\Omega \nabla u\cdot\nabla v - \int_{\partial\Omega} gv + \int_\Omega quv = \int_\Omega fv \quad \longrightarrow$$

$$a(u,v) := \int_\Omega \nabla u\cdot\nabla v + \int_\Omega quv = \int_\Omega fv + \int_{\partial\Omega} gv =: F(v)$$

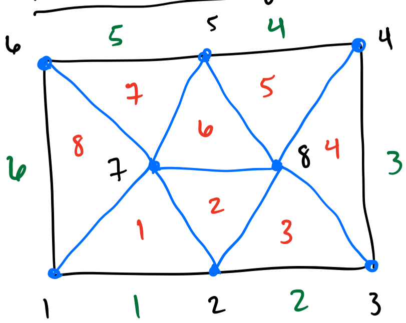
$$u = \sum_i u_i \phi_i$$

$$v = \phi_j \quad \rightarrow \quad \sum_i a(\phi_i, \phi_j)\, u_i = F(\phi_j)$$

$$\Big[a(\phi_i,\phi_j)\Big]_{\underset{\substack{\uparrow\\ \text{row}}}{j},\underset{\substack{\uparrow\\ \text{col}}}{i}} \Big[u_i\Big]_{\underset{\substack{\uparrow\\ \text{row}}}{i}} = \Big[F(\phi_j)\Big]_{\underset{\substack{\uparrow\\ \text{row}}}{j}}$$

$$a(\phi_i, \phi_j) = \int_\Omega \nabla \phi_i \cdot \nabla \phi_j + \int_\Omega q\, \phi_i \phi_j$$

Enumerating Everything (Example)



1. Enumerate nodes

$$\begin{bmatrix} (x_1, y_1) \\ (x_2, y_2) \\ \vdots \\ (x_8, y_8) \end{bmatrix}$$

2. Enumerate Elements

$$\begin{array}{cc} & \ell \\ k & \begin{bmatrix} 1 & 2 & 7 \\ 2 & 8 & 7 \\ 2 & 3 & 8 \\ 3 & 4 & 8 \\ 4 & 5 & 8 \\ 5 & 7 & 8 \\ 5 & 6 & 7 \\ 1 & 7 & 8 \end{bmatrix} \end{array}$$

element 1 is composed of nodes 1, 2, and 7 in the nodes list. It's vertices are (x_1, y_1), (x_2, y_2), (x_7, y_7).

* This defines a map $I(\ell, m)$

ℓ – element #

m – local vertex # (1, 2, 3)

$I(\ell, m)$ – global vertex #

3. Enumerate Boundary Edges

$$\begin{array}{cc} & m \\ \ell & \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \\ 4 & 5 \\ 5 & 6 \\ 6 & 1 \end{bmatrix} \end{array}$$

edge 1 has vertices (x_1, y_1) and (x_2, y_2)

* Defines a map $J(\ell, m)$

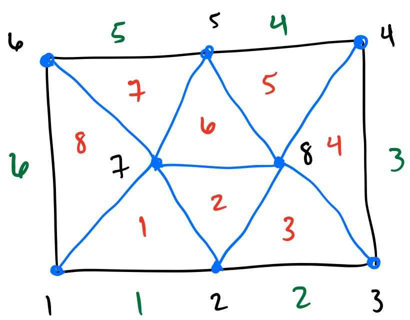
ℓ – edge #

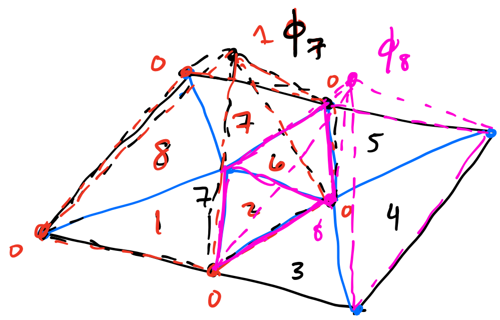
m – local vertex # (1 or 2)

$J(\ell, m)$ – global vertex #

Element-wise assembly

$$a(\phi_i, \phi_j) = \int_\Omega \nabla\phi_i \cdot \nabla\phi_j + \int_\Omega q\,\phi_i\,\phi_j$$





- $\operatorname{supp}\phi_i$ is a union of cells

  Eg $\operatorname{supp}\phi_7 = K_1 \cup K_2 \cup K_6 \cup K_7 \cup K_8$

  $\operatorname{supp}\phi_8 = K_2 \cup K_3 \cup K_4 \cup K_5 \cup K_6$

- $\operatorname{supp}\phi_i \cap \operatorname{supp}\phi_j$ is either a set of measure 0 (basically empty for our purposes) or it is a union of cells

  Eg $\operatorname{supp}\phi_7 \cap \operatorname{supp}\phi_8 = K_2 \cup K_6$

- Therefore, $a(\phi_i, \phi_j) = 0$ or

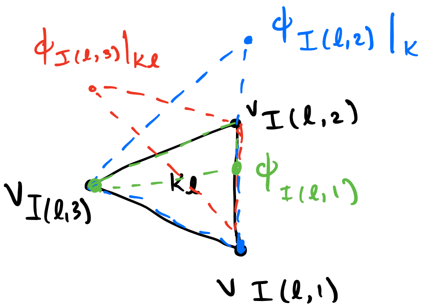
$$a(\phi_i, \phi_j) = \sum_{K \subset \operatorname{supp}\phi_i \cap \operatorname{supp}\phi_j} \underbrace{\int_K \nabla\phi_i \cdot \nabla\phi_j + \int_K q\,\phi_i\,\phi_j}_{:= a_K(\phi_i, \phi_j)}$$

eg $a(\phi_7, \phi_8) = a_2(\phi_7, \phi_8) + a_6(\phi_7, \phi_8)$.

- Calculating $a(\phi_i, \phi_j)$ involves calculating $a_\ell(\phi_i, \phi_j)$ where ℓ varies over the element numbers

- Consider this procedure

1. Fix an element K_ℓ



2. The <u>only</u> basis functions ϕ_i where $K_\ell \subset \operatorname{supp} \phi_i$ are

   $\phi_{I(\ell,1)}, \; \phi_{I(\ell,2)}, \; \phi_{I(\ell,3)}$

3. Thus the <u>only</u> $a_\ell(\phi_i, \phi_j)$ that are <u>not</u> zero are $a_\ell(\phi_{I(\ell,m)}, \phi_{I(\ell,n)})$ for $m, n = 1,2,3$

4. For each $m, n = 1,2,3$:

   calculate $a_\ell(\phi_{I(\ell,m)}, \phi_{I(\ell,n)})$ and <u>add</u> it to the entry corresponding to $a(\phi_{I(\ell,m)}, \phi_{I(\ell,n)})$ in $\left[a(\phi_i, \phi_j)\right]_{j,i}$ matrix

Claim: If we do step 4 for each element K_ℓ, then we will have completely assembled the matrix $\left[a(\phi_i, \phi_j)\right]_{j,i}$

Example

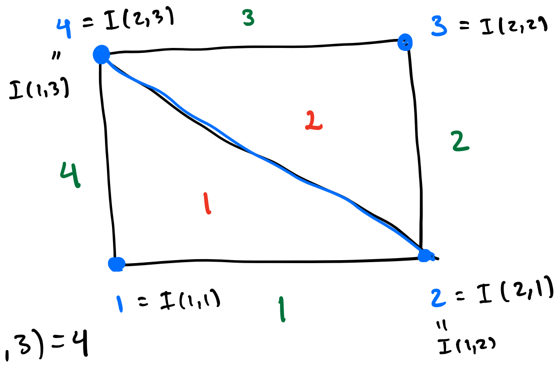
$I(1,1) = 1$

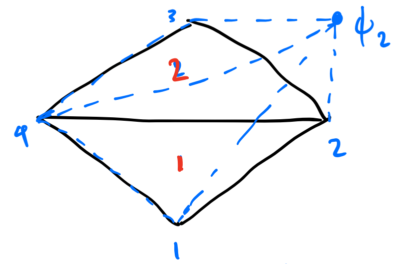
$I(1,2) = 2$

$I(1,3) = 4$

$I(2,1) = 2$

$I(2,2) = 3$ $\quad I(2,3) = 4$





$$a(\phi_2,\phi_2) = a_1(\phi_2,\phi_2) + a_2(\phi_2,\phi_2)$$

$$= a_1(\phi_{I(1,2)},\phi_{I(1,2)}) + a_2(\phi_{I(2,1)},\phi_{I(2,1)})$$

$$\begin{bmatrix} a(\phi_1,\phi_1) & a(\phi_2,\phi_1) & a(\phi_3,\phi_1) & a(\phi_4,\phi_1) \\ a(\phi_1,\phi_2) & \boxed{a(\phi_2,\phi_2)} & a(\phi_3,\phi_2) & a(\phi_4,\phi_2) \\ a(\phi_1,\phi_3) & a(\phi_2,\phi_3) & a(\phi_3,\phi_3) & a(\phi_4,\phi_3) \\ a(\phi_1,\phi_4) & a(\phi_2,\phi_4) & a(\phi_3,\phi_4) & a(\phi_4,\phi_4) \end{bmatrix} =$$

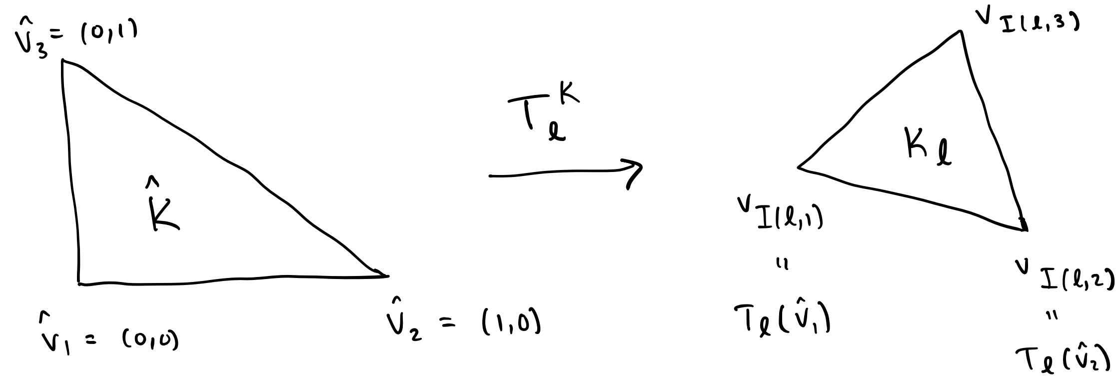
$$\begin{bmatrix} a_1(\phi_1,\phi_1) & a_1(\phi_2,\phi_1) & 0 & a_1(\phi_4,\phi_1) \\ a_1(\phi_1,\phi_2) & \boxed{a_1(\phi_2,\phi_2)} & 0 & a_1(\phi_4,\phi_2) \\ 0 & 0 & 0 & 0 \\ a_1(\phi_1,\phi_4) & a_1(\phi_2,\phi_4) & {} & a_1(\phi_4,\phi_4) \\ 0 & 0 & 0 & 0 \end{bmatrix} +$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ {} & \boxed{a_2(\phi_2,\phi_2)} & a_2(\phi_3,\phi_2) & a_2(\phi_4,\phi_2) \\ 0 & a_2(\phi_2,\phi_3) & a_2(\phi_3,\phi_3) & a_2(\phi_4,\phi_3) \\ 0 & a_2(\phi_2,\phi_4) & a_2(\phi_3,\phi_4) & a_2(\phi_4,\phi_4) \end{bmatrix}$$

Calculating $a_\ell(\phi_{I(\ell,m)}, \phi_{I(\ell,n)})$:

$$a_\ell(\phi_{I(\ell,m)}, \phi_{I(\ell,n)}) = \underbrace{\int_{K_\ell} \nabla\phi_{I(\ell,m)} \cdot \nabla\phi_{I(\ell,n)}\, dx}_{:= S(\ell,m,n) \text{ stiffness term}} + \underbrace{\int_{K_\ell} q\, \phi_{I(\ell,m)}\, \phi_{I(\ell,n)}\, dx}_{:= M(\ell,m,n) \text{ mass term}}$$

Reference Maps!

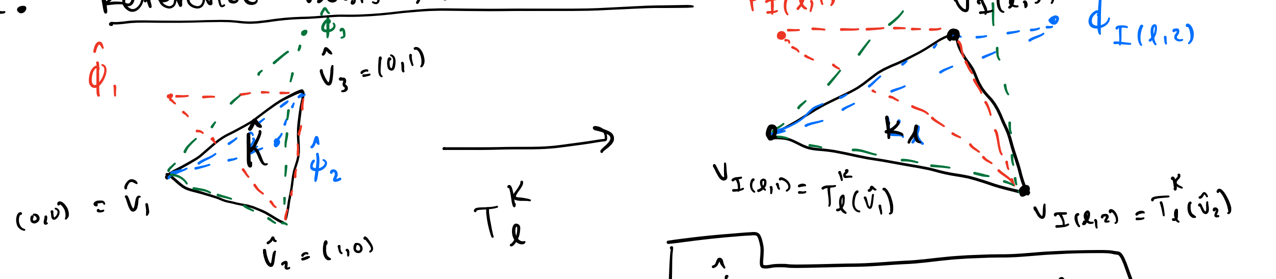


$$T_\ell^K(\hat{x}) := v_{I(\ell,1)} + (v_{I(\ell,2)} - v_{I(\ell,1)})\,\hat{x}_1 + (v_{I(\ell,3)} - v_{I(\ell,1)})\,\hat{x}_2$$

$$= v_{I(\ell,1)} + \underbrace{\left[\, v_{I(\ell,2)} - v_{I(\ell,1)} \;\middle|\; v_{I(\ell,3)} - v_{I(\ell,1)} \,\right]}_{\text{matrix w/ columns } := B_\ell} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix}$$

Facts about T_ℓ^K : 1. $\underbrace{DT_\ell^K}_{\text{Jacobian matrix}} = B_\ell$

2. Reference basis functions on $\hat{K}$



$$\boxed{(*)\quad \phi_{I(\ell,m)} \circ T_\ell^K = \hat\phi_m^K \qquad m = 1,2,3}$$

$$\boxed{\begin{aligned} \hat\phi_1^K(\hat x) &= 1 - \hat x_1 - \hat x_2 \\ \hat\phi_2^K(\hat x) &= \hat x_1 \\ \hat\phi_3^K(\hat x) &= \hat x_2 \end{aligned}}$$

$$M(\ell,m,n) = \int_{K_\ell} q(x)\, \phi_{I(\ell,m)}(x)\, \phi_{I(\ell,n)}(x)\, dx$$

Change variables $x = T_\ell(\hat x)$ $\longrightarrow$ $=$

comes from changing variables in multiple dimensions

$$\int_{\hat K} q \circ T_\ell^K(\hat x) \cdot \underbrace{\phi_{I(\ell,m)} \circ T_\ell^K(\hat x)}_{\hat\phi_m^K(\hat x)} \cdot \underbrace{\phi_{I(\ell,n)} \circ T_\ell^K(\hat x)}_{\hat\phi_n^K(\hat x)} \cdot \left|\det DT_\ell^K\right| d\hat x$$

$$\boxed{M(\ell,m,n) = \left|\det B_\ell\right| \int_{\hat K} q \circ T_\ell^K\, \hat\phi_m^K\, \hat\phi_n^K\, d\hat x}$$

much simpler to calculate!

$$S(\ell,m,n) = \int_K \nabla_x \phi_{I(\ell,m)}(x) \cdot \nabla_x \phi_{I(\ell,n)}(x)\, dx$$

$$= \int_{\hat{K}} (\nabla_x \phi_{I(l,m)}) \circ T_l^K(\hat{x}) \cdot (\nabla_x \phi_{I(l,n)}) \circ T_l^K(\hat{x}) \, |\det DT_l^K| \, d\hat{x}$$

change variables $x = T_l^K(\hat{x})$

Chain rule:

$$\underbrace{\nabla_{\hat{x}}}_{\substack{\text{gradient} \\ \text{wrt } \hat{x} \text{ variables}}} \big(\underbrace{\phi_{I(l,m)} \circ T_l^K}_{\hat{\phi}_m} \big)(\hat{x}) = \underbrace{(\nabla_x}_{\text{gradient wrt x variables}} \phi_{I(l,m)}) \circ T_l^K(\hat{x}) \; DT_l^K$$

$$= (\nabla_x \phi_{I(l,m)}) \circ T_l^K(\hat{x}) \; B_l$$

$$\rightarrow \boxed{\nabla_{\hat{x}} \hat{\phi}_m^K(\hat{x}) B_l^{-1} = (\nabla_x \phi_{I(l,m)}) \circ T_l^K(\hat{x}) \qquad m = 1,2,3}$$

$$S(l,m,n) = |\det B_l| \int_{\hat{K}} \Big(\underbrace{\underbrace{\nabla_{\hat{x}} \hat{\phi}_m^K(\hat{x})}_{\text{row vector}} \, \underbrace{B_l^{-1}}_{\text{matrix}}}_{\text{row vector}} \Big) \underbrace{\cdot}_{\text{dot product of 2 vectors}} \Big(\nabla_{\hat{x}} \hat{\phi}_n^K(\hat{x}) B_l^{-1} \Big) \, d\hat{x}$$

$$\boxed{\begin{array}{ll} \nabla_{\hat{x}} \hat{\phi}_1^K = (-1,-1) & \\ \nabla_{\hat{x}} \hat{\phi}_2^K = (1,0) & \underline{\text{constant}}\,! \\ \nabla_{\hat{x}} \hat{\phi}_3^K = (0,1) & \end{array}} \rightarrow$$

$$S(l,m,n) = |\det B_l| \underbrace{\text{area}(\hat{K})}_{\frac{1}{2}} (\nabla \hat{\phi}_m^K B_l^{-1}) \cdot (\nabla \hat{\phi}_n^K B_l^{-1})$$

$$\boxed{S(l,m,n) = \frac{|\det B_l|}{2} (\nabla \hat{\phi}_m^K B_l^{-1}) \cdot (\nabla \hat{\phi}_n^K B_l^{-1})}$$

No integral needed!

Approximate Integrals w/ Quadrature Rules

Quadrature on reference triangle idea

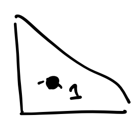
$f: \hat{K} \to \mathbb{R}$, $\hat{x}_1^K, \dots, \hat{x}_n^K \in \hat{K}$, $\hat{w}_1^K, \dots, \hat{w}_n^K \in \mathbb{R}$

$$\int_{\hat{K}} f(\hat{x})\, d\hat{x} \approx \text{area}(\hat{K}) \sum_{i=1}^{n} \hat{w}_i^K f(\hat{x}_i^K) = \frac{1}{2} \sum_{i=1}^{n} \hat{w}_i^K f(\hat{x}_i^K)$$

Some good choices for $\{\hat{x}_i^K\}$, $\{\hat{w}_i^K\}$:

1. Centroid, exact for when f is degree 1 or less

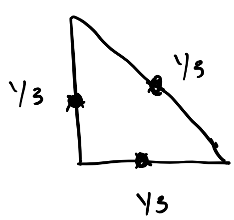
   $n=1$, $\hat{x}_1^K = (1/3, 1/3)$, $\hat{w}_1^K = 1$



2. midpoint, exact for when f is degree 2 or less

   $n = 3$ $\hat{x}_1^K = (1/2, 0)$, $\hat{x}_2^K = (1/2, 1/2)$, $\hat{x}_3^K = (0, 1/2)$

   $\hat{w}_1^K = \hat{w}_2^K = \hat{w}_3^K = 1/3$



Can find higher order quadrature rules online.

$$\boxed{M(l,m,n) \approx \frac{|\det B_K|}{2} \sum_i \hat{w}_i^K\, q_r(T_l^K(\hat{x}_i^K))\, \hat{\phi}_m^K(\hat{x}_i^K)\, \hat{\phi}_n^K(\hat{x}_i^K)}$$

Altogether:

$$a_\ell(\phi_{I(\ell,m)}, \phi_{I(\ell,n)}) \approx \overbrace{\frac{|\det B_\ell|}{2} (\nabla \hat{\phi}_m^K B_\ell^{-1}) \cdot (\nabla \hat{\phi}_n^K B_\ell^{-1})}^{S(\ell,m,n)} +$$

$$\underbrace{\frac{|\det B_\ell|}{2} \sum_{i=1}^{\text{number quadrature points}} \hat{w}_i^K q(T_\ell^K(\hat{x}_i^K)) \hat{\phi}_m^K(\hat{x}_i^K) \hat{\phi}_n^K(\hat{x}_i^K)}_{\approx M(\ell,m,n)}.$$

Computable!

We can now assemble the matrix $[a(\phi_i, \phi_j)]_{j,i}$.

$$F(\phi_j) = \underbrace{\int_\Omega f\phi_j}_{F_\Omega(\phi_j)} + \underbrace{\int_{\partial\Omega} g\phi_j}_{F_{\partial\Omega}(\phi_j)} \longrightarrow$$

$$[F(\phi_j)]_j = [F_\Omega(\phi_j)]_j + [F_{\partial\Omega}(\phi_j)]_j$$

Assembling $[F_\Omega(\phi_j)]$ is similar to $[a(\phi_i, \phi_j)]_{j,i}$

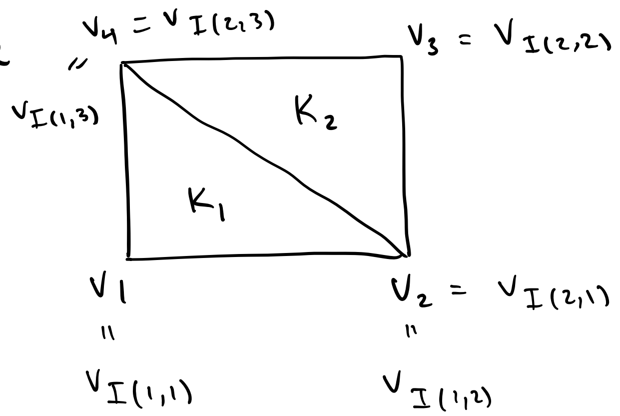
$$F_\Omega(\phi_j) = \sum_{K_\ell \subset \operatorname{supp}\phi_j} \underbrace{\int_{K_\ell} f\phi_j}_{:= F_{\Omega,\ell}(\phi_j)} \longrightarrow$$

For each element K_ℓ, calculate

$\int_{K_\ell} f \phi_{I(\ell,m)}$ and add it to the entry corresponding

to $F_\Omega(\phi_{I(\ell,m)})$ in $[F(\phi_j)]_j$

Example



$$\begin{bmatrix} F_\Omega(\phi_1) \\ F_\Omega(\phi_2) \\ F_\Omega(\phi_3) \\ F_\Omega(\phi_4) \end{bmatrix} = \begin{bmatrix} F_{\Omega,1}(\phi_{I(1,1)}) \\ F_{\Omega,1}(\phi_{I(1,2)}) \\ 0 \\ F_{\Omega,1}(\phi_{I(1,3)}) \end{bmatrix} + \begin{bmatrix} 0 \\ F_{\Omega,2}(\phi_{I(2,1)}) \\ F_{\Omega,2}(\phi_{I(2,2)}) \\ F_{\Omega,2}(\phi_{I(2,3)}) \end{bmatrix}$$

Calculating $F_{\Omega,\ell}(\phi_{I(\ell,m)})$:

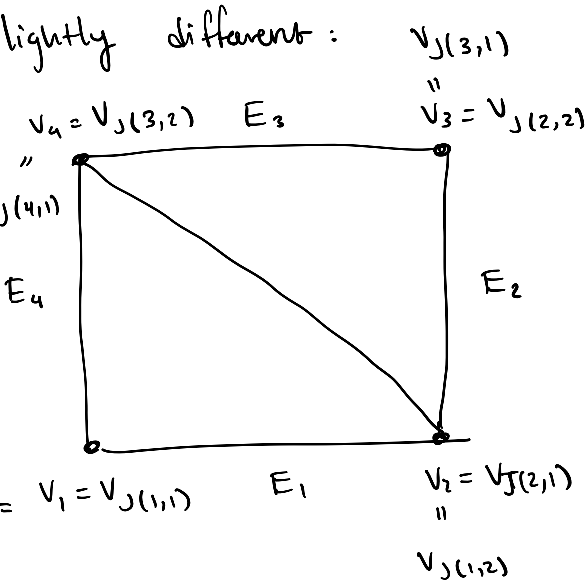
$$\int_{K_\ell} f \phi_{I(\ell,m)} \, dx = \int_{\hat{K}} f \circ T_\ell^K \hat{\phi}_m \, |\det DT_\ell^K| \, d\hat{x}$$

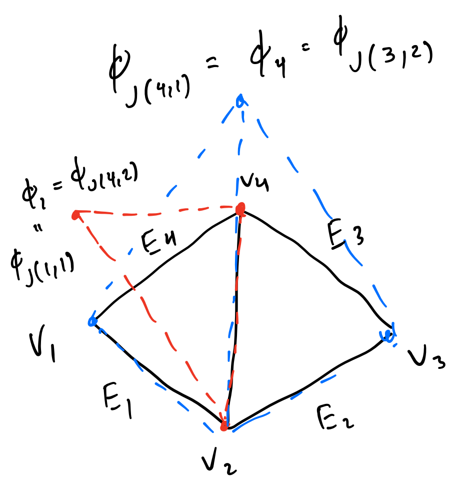
$$= |\det B_\ell| \int_{\hat{K}} f \circ T_\ell^K \hat{\phi}_m^K \approx \boxed{\frac{|\det B_\ell|}{2} \sum_{i=1}^{\#\text{quad pts}} \hat{w}_i^K f(T_\ell^K(\hat{x}_i^K)) \hat{\phi}_m^K(\hat{x}_i^K)}$$

Assembling $[F_{\partial\Omega}(\phi_j)]_j$ is slightly different:

$$F_{\partial\Omega}(\phi_j) = \int_{\partial\Omega} g\phi_j$$

Example



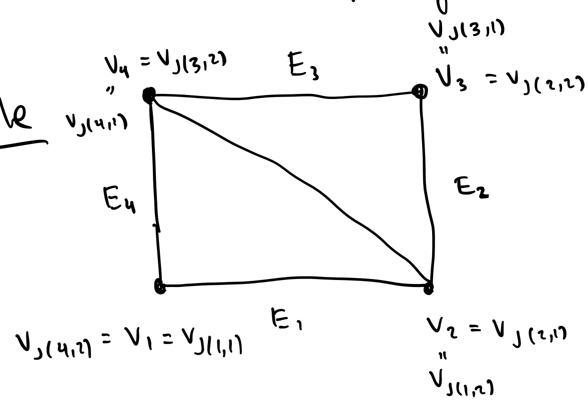


$$F_{\partial\Omega}(\phi_j) = \sum_{E_\ell \subset \mathrm{supp}\,\phi_j} \underbrace{\int_{E_\ell} g\phi_j}_{F_{\partial\Omega,\ell}(\phi_j)}$$

The only nonzero $F_{\partial\Omega,\ell}(\phi_j)$ are $F_{\partial\Omega,\ell}(\phi_{J(\ell,m)})$ $m=1,2$.

Thus for each edge E_ℓ, if we calculate $F_{\partial\Omega,\ell}(\phi_{J(\ell,m)})$ and add it to the corresponding entry in $[F_{\partial\Omega}(\phi_j)]_j$ then we will have completely assembled $[F_{\partial\Omega}(\phi_j)]_j$

Example

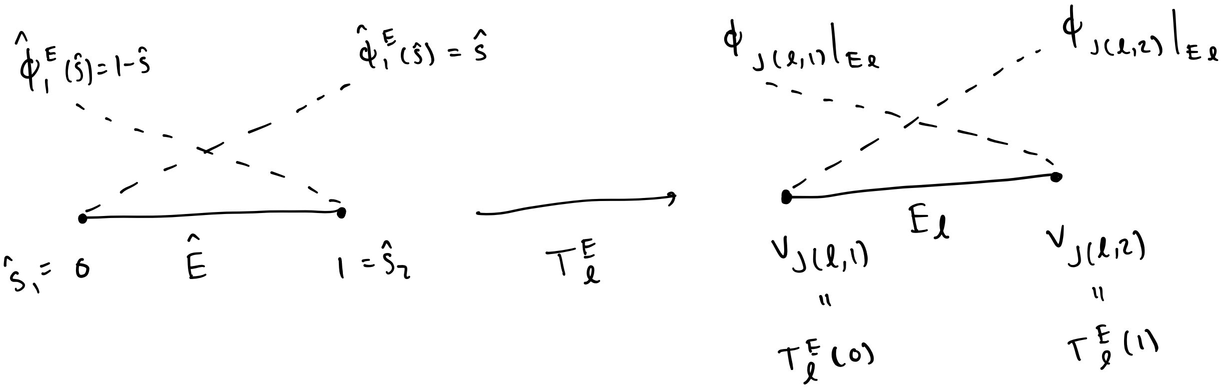


$$\begin{bmatrix} F_{\partial\Omega}(\phi_1) \\ F_{\partial\Omega}(\phi_2) \\ F_{\partial\Omega}(\phi_3) \\ F_{\partial\Omega}(\phi_4) \end{bmatrix} = \begin{bmatrix} F_{\partial\Omega,1}(\phi_{J(1,1)}) \\ F_{\partial\Omega,1}(\phi_{J(1,2)}) \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ F_{\partial\Omega,2}(\phi_{J(2,1)}) \\ F_{\partial\Omega,2}(\phi_{J(2,2)}) \\ 0 \end{bmatrix} +$$

$$\begin{bmatrix} 0 \\ 0 \\ F_{\partial\Omega,3}(\phi_{J(3,1)}) \\ F_{\partial\Omega,3}(\phi_{J(3,2)}) \end{bmatrix} + \begin{bmatrix} F_{\partial\Omega,4}(\phi_{J(4,2)}) \\ 0 \\ 0 \\ F_{\partial\Omega,4}(\phi_{J(4,1)}) \end{bmatrix}$$

Calculating $F_{\partial\Omega,\ell}(\phi_{J(\ell,m)})$:

Need reference maps for <u>edges</u> !



$$T_\ell^E(\hat s) = V_{J(\ell,1)} + \left(V_{J(\ell,2)} - V_{J(\ell,1)}\right)\hat s \longrightarrow \frac{d}{d\hat s} T_\ell^E(\hat s) = \underbrace{V_{J(\ell,2)} - V_{J(\ell,1)}}_{\text{is a constant vector!}}$$

$$\phi_{J(\ell,m)}\big|_{E_\ell} \circ T_\ell^E(\hat s) = \hat\phi_m^E(\hat s) \qquad m = 1,2$$

$$F_{\partial\Omega,\ell}\left(\phi_{J(\ell,m)}\right) = \int_{E_\ell} g\,\phi_{J(\ell,m)}\,ds$$

$$s = T_\ell^E(\hat{s}) \longrightarrow \quad = \int_{\hat{E}} g\circ T_\ell^E\,\hat{\phi}_m^E\,\underbrace{\left|\frac{d}{d\hat{s}}T_\ell^E(\hat{s})\right|}\,d\hat{s}$$

Euclidean norm of this 2-vector. Comes from changing variables in multiple dimensions.

$$= \left|V_{J(\ell,2)} - V_{J(\ell,1)}\right| \underbrace{\int_{\hat{E}} g\circ T_\ell^E\,\hat{\phi}_m^E\,d\hat{s}}$$

Need 1D quadrature rules to approximate this

Good 1D quadrature rules : (Gauss Quadrature)

$$\int_0^1 f(x)\,dx \approx \frac{1}{2}\sum_{i=1}^{\#qP} \hat{w}_i^E\, f\left(\frac{1}{2}\hat{x}_i^E + \frac{1}{2}\right)$$

1. $\#qp = 1$, $\hat{x}_1^E = 0$, $\hat{w}_1^E = 2$. Exact for f a deg ≤ 1 poly.

2. $\#qp = 2$, $\hat{x}_1^E = -\frac{1}{\sqrt{3}}$, $\hat{x}_2^E = \frac{1}{\sqrt{3}}$, $\hat{w}_1^E = \hat{w}_2^E = 1$
Exact for degree ≤ 3 polynomials.

Can find higher order rules online.

Thus

$$\boxed{F_{\partial\Omega,\ell}(\phi_{J(\ell,m)}) \approx \frac{|V_{J(\ell,2)} - V_{J(\ell,1)}|}{2} \sum_{i=1}^{\#qp} \hat{w}_i^E \, g\left(T_\ell^E\left(\tfrac{1}{2}\hat{x}_i^E + \tfrac{1}{2}\right)\right) \hat{\phi}_m^E\left(\tfrac{1}{2}\hat{x}_i^E + \tfrac{1}{2}\right)}$$

We can now completely assemble our system

$$\left[a(\phi_i, \phi_j)\right]_{j,i} \left[u_i\right]_i = \left[F_\Omega(\phi_j)\right]_j + \left[F_{\partial\Omega}(\phi_j)\right]_j$$

and solve for $[u_i]_i$.