Proof of Strang's First Lemma

Let $(V, \|\cdot\|)$ be a Banach space. Let $a(\cdot,\cdot)$ be a continuous and coercive bilinear form on V. Let $F(\cdot)$ be a continuous linear form on V. Let u be the unique solution to $a(u,v) = F(v)$ for all v in V. For each $h > 0$, let $V_h \subset V$ be a subspace of V, let $a_h(\cdot,\cdot)$ be a continuous bilinear form on V_h, and let $F_h(\cdot)$ be a continuous linear form on V_h. Suppose that the family $(a_h)_h$ is uniformly V_h-elliptic: there is a constant $\bar{\alpha} > 0$ st for all h, $\bar{\alpha}\|v_h\|^2 \le a_h(v_h, v_h)$ for all $v_h \in V_h$.

Let u_h be the unique solution in V_h to $a_h(u_h, v_h) = F_h(v_h)$ for all v_h in V_h. Then for any $v_h \in V_h$, $\|u-u_h\| \le \|u-v_h\| + \|v_h-u_h\|$. By the uniform V_h-ellipticity,

$$\bar{\alpha}\|v_h-u_h\|^2 \le a_h(v_h-u_h, v_h-u_h)$$

$$= a_h(v_h, v_h-u_h) - \underbrace{a_h(u_h, v_h-u_h)}_{F_h(v_h-u_h)}$$

$$= a(v_h, v_h-u_h) + a_h(v_h, v_h-u_h) - a(v_h, v_h-u_h) + F(v_h-u_h) - F_h(v_h-u_h) - \underbrace{F(v_h-u_h)}_{a(u,\, v_h-u_h)}$$

$$\begin{pmatrix} \pm\, a(v_h, v_h-u_h) \\ \pm\, F(v_h-u_h) \end{pmatrix}$$

$$= \underline{a(v_h - u, v_h - u_h)} + a_h(v_h, v_h - u_h) - a(v_h, v_h - u_h)$$

$$+ F(v_h - u_h) - F_h(v_h - u_h)$$

$$\leq |a(v_h - u, v_h - u_h)|$$

$$+ \frac{|a_h(v_h, v_h - u_h) - a(v_h, v_h - u_h)|}{\|v_h - u_h\|} \cdot \|v_h - u_h\|$$

$$+ \frac{|F(v_h - u_h) - F_h(v_h - u_h)|}{\|v_h - u_h\|} \cdot \|v_h - u_h\| \longrightarrow$$

continuity constant of a

$$\bar{\alpha} \|v_h - u_h\|^2 \leq C_a \|v_h - u\| \|v_h - u_h\|$$

$$+ \underbrace{\sup_{w_h \in V_h} \frac{|a_h(v_h, w_h) - a(v_h, w_h)|}{\|w_h\|}}_{:= C_{v_h}} \|v_h - u_h\|$$

$$+ \underbrace{\sup_{w_h \in V_h} \frac{|F(w_h) - F_h(w_h)|}{\|w_h\|}}_{:= C_{F_h}} \|v_h - u_h\| \longrightarrow$$

(divide $\bar{\alpha} \|v_h - u_h\|$)

$$\downarrow$$

$$\|v_h - u_h\| \leq \frac{C_a}{\bar{\alpha}} \|v_h - u\| + \frac{1}{\bar{\alpha}} C_{v_h} + \frac{1}{\bar{\alpha}} C_{F_h} \longrightarrow$$

$$\|u-u_h\| \le \|u-v_h\| + \|v_h-u_h\| \le \|u-v_h\| + \frac{C_a}{\alpha}\|v_h-u\| + \frac{C_{v_h}}{\alpha} + \frac{C_{F_h}}{\alpha}$$

$$\rightarrow \quad \|u-u_h\| \le \left(1+\frac{C_a}{\alpha}\right)\|u-v_h\| + \frac{1}{\alpha}C_{v_h} + \frac{1}{\alpha}C_{F_h}$$

$$\le \underbrace{\max\left(1+\frac{C_a}{\alpha}, \frac{1}{\alpha}\right)}_{:= C}\left(\|u-v_h\| + C_{v_h} + C_{F_h}\right) \quad \text{for all } v_h \in V_h$$

Since C is independent of h, we conclude that

$$\boxed{\begin{array}{l}\text{There exists } C>0 \text{ st for all } h, \\ \|u-u_h\| \le C\left[\inf_{v_h \in V_h}\left(\|u-v_h\| + \sup_{w_h \in V_h}\frac{|a_h(v_h,w_h) - a(v_h,w_h)|}{\|w_h\|}\right) + \sup_{w_h \in V_h}\frac{|F(w_h) - F_h(w_h)|}{\|w_h\|}\right] \quad \square\end{array}}$$