# MATH 610 Homework 1 Hints

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# 1 Exercise 1

## 1.1 Problem 1

A function $u$ belongs to $H^1(-1,1)$ if and only if

1. $u$ belongs to $L^2(-1,1)$,
2. $u$ has a weak derivative in $L^2(-1,1)$.

A function $u$ belongs to $L^2(-1,1)$ if and only if the integral

$$\int_{-1}^{1} u(x)^2 \, \mathrm{d}x$$

exists and is finite. One way to show this is to explicitly compute the integral. There is a more elegant way to do this without computing anything. I'll let you figure that one out.

To find a weak derivative of $u$, let $\varphi$ be a test function, meaning that $\varphi \in C_0^\infty([-1,1])$, which means that

1. $\varphi$ is infinitely differentiable,
2. $\varphi(-1) = 0$,
3. $\varphi(1) = 0$.

Then split up the integral over the pieces of $u$

$$\int_{-1}^{1} u(x)\varphi'(x) \, \mathrm{d}x = \int_{-1}^{0} u(x)\varphi'(x) \, \mathrm{d}x + \int_{0}^{1} u(x)\varphi'(x) \, \mathrm{d}x$$

and then do integration by parts. See what falls out at the end to find a candidate for the weak derivative $v$ of $u$, and then check if $v \in L^2(-1,1)$.

## 1.2 Problem 2

This is a generalization of problem 1, so we proceed similarly. This time, explicitly compute the integral of $u(x)^2$ where $u(x) = |x|^\alpha$ and see which values of $\alpha$ give you a finite integral. This tells you which $\alpha$ allows $u \in L^2(a,b)$. Then let $\varphi$ be a test function, do integration by parts over the pieces of $u$ as in problem 1, and see what falls out to give you a candidate for the weak $v$ derivative of $u$. This will tell you which $\alpha$ allows for the integration-by-parts to happen at all and thus give you a candidate. Then compute the integral of $v(x)^2$ to see which $\alpha$ allow for $v \in L^2(-1,1)$.

## 1.3 Problem 3

Be careful here. Observe that if $\varphi$ is a test function on $[-1,1]$, this does not mean that the restriction $\varphi|_{[-1,0]}$ is a test function on $[-1,0]$, nor is it a test function when restricted to $[0,1]$. Therefore, we cannot immediately do the following calculation

$$\begin{aligned}\int_{-1}^{1} u(x)\varphi'(x)\,\mathrm{d}x &= \int_{-1}^{0} u_1(x)\varphi'(x)\,\mathrm{d}x + \int_{0}^{1} u_2(x)\varphi'(x)\,\mathrm{d}x \\ &= -\int_{-1}^{0} u_1'(x)\varphi(x)\,\mathrm{d}x - \int_{0}^{1} u_2'(x)\varphi(x)\,\mathrm{d}x\end{aligned}$$

since we can only go to the second line when $\varphi|_{[-1,0]}$ is a test function on $[-1,0]$ and $\varphi|_{[0,1]}$ is a test function on $[0,1]$. We have to be a bit more clever here to justify this. To start, we need the following theorem.

**Theorem 1.** *If $u \in H^1(a,b)$, then there is a sequence $u_n$ of smooth functions in $C^\infty(a,b)$ that converges to $u$ in $H^1(a,b)$.*

We apply this to $u_1$ and $u_2$ to get a sequence $u_n^1$ in $H^1(-1,0)$ and a sequence $u_n^2$ in $H^1(0,1)$ where $u_n^i$ converges to $u_i$. Let

$$u_n(x) = \begin{cases} u_n^1(x) & x \in (-1,0) \\ u_n^2(x) & x \in [0,1) \end{cases}$$

Then we have that

$$\begin{aligned}\int_{-1}^{1} u_n(x)\varphi'(x)\,\mathrm{d}x &= \int_{-1}^{0} u_n^1(x)\varphi'(x)\,\mathrm{d}x + \int_{0}^{1} u_n^2(x)\varphi'(x)\,\mathrm{d}x \\ &= -\int_{-1}^{0} (u_n^1)'(x)\varphi(x)\,\mathrm{d}x - \int_{0}^{1} (u_n^2)'(x)\varphi(x)\,\mathrm{d}x + (u_n^1(0) - u_n^2(0))\varphi(0) \\ &= -\int_{-1}^{0} v_n(x)\varphi(x)\,\mathrm{d}x + (u_n^1(0) - u_n^2(0))\varphi(0)\end{aligned}$$

where the second step follows from integration-by-parts, which is allowed now because the $u_n^i$ are classically smooth and not only in $H^1$, and

$$v_n(x) = \begin{cases} (u_n^1)'(x) & x \in (-1, 0) \\ (u_n^2)'(x) & x \in [0, 1) \end{cases}.$$

Set

$$v(x) = \begin{cases} u_1'(x) & x \in (-1, 0) \\ u_2'(x) & x \in [0, 1) \end{cases}.$$

Show that

$$\begin{aligned} \int_{-1}^{1} u_n(x)\varphi'(x)\,\mathrm{d}x &\to \int_{-1}^{1} u(x)\varphi'(x)\,\mathrm{d}x, \\ \int_{-1}^{1} v_n(x)\varphi(x)\,\mathrm{d}x &\to \int_{-1}^{1} v(x)\varphi(x)\,\mathrm{d}x, \\ (u_n^1(0) - u_n^2(0))\varphi(0) &\to 0, \\ v &\in L^2(-1, 1) \end{aligned}$$

as $n \to \infty$ and conclude that $u \in H^1(-1, 1)$ with $v$ as its weak derivative. You are free to use the following theorem, which is a consequence of the optional problem 5.

**Theorem 2.** *Fix $x_0 \in [a, b]$. Then the map*

$$E_{x_0}(u) = u(x_0)$$

*is a continuous linear functional on $H^1(a, b)$. In other words,*

$$E_{x_0}(cu + v) = cE_{x_0}(u) + E_{x_0}(v)$$

*for all $u, v \in H^1(a, b)$ and all $c \in \mathbb{R}$, and there is a constant $C > 0$ such that*

$$|E_{x_0}(u)| \leq C\|u\|_{H^1(a,b)}$$

*for all $u \in H^1(a, b)$.*

## 1.4 Problem 4

Let $u \in H^1(a, b)$. Then there is a sequence $v_n$ of smooth functions in $C^\infty(a, b)$ that converges to $u$ in $H^1(a, b)$. Use the fact that

$$\|v\|_{L^\infty(a,b)} \leq C\|v\|_{H^1(a,b)}$$

when $v \in C^\infty(a, b)$ (which is a consequence of the last inequality in question 2 of exercise 2) to argue that $v_n$ is Cauchy in $L^\infty(a, b)$. Since $L^\infty(a, b)$ is complete, this implies that there exists $v \in L^\infty(a, b)$ such that $v_n \to v$ in $L^\infty(a, b)$. Now argue that $u = v$.

# 2 Exercise 2

## 2.1 Problem 1

Use the triangle inequality and the Cauchy-Schwarz inequality

$$\begin{aligned} |u(x)| &\leq |u(y)| + \int_0^1 |u'(s)| \, \mathrm{d}s \\ &= |u(y)| + (1, |u'|)_{L^2(0,1)} \\ &\leq |u(y)| + \|u'\|_{L^2(0,1)}. \end{aligned}$$

Now pick particular points for $y$.

## 2.2 Problem 2

For the first inequality, first integrate

$$u(x) = \int_0^1 u(y) \, \mathrm{d}y + \int_0^1 \int_y^x u'(s) \, \mathrm{d}s \, \mathrm{d}y.$$

Then use the triangle inequality and Cauchy-Schwarz:

$$|u(x)| \leq |\overline{u}| + \|u'\|_{L^2(0,1)}$$

where

$$\overline{u} = \int_0^1 u(y) \, \mathrm{d}y.$$

At some point you will need to use Young's inequality:

$$2ab \leq a^2 + b^2,$$

which follows from the fact that

$$(a - b)^2 \geq 0$$

for all $a, b \in \mathbb{R}$.

For the second and third inequalities, do essentially the same thing as problem 1, pick particular points for $y$, and apply Young's inequality.

For the last inequality, proceed as in problem 1 to get

$$|u(x)| \leq |u(y)| + \|u'\|_{L^2(0,1)}.$$

Then square, apply Young's inequality, and integrate.