MATH 610 Homework 2 Hints

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1 Exercise 1

1.1 Problem 1

Suppose that you have a smooth function u that satisfies the boundary conditions $u(0) = u(1) = 0$ and which solves the ODE

$$-(ku')' + bu' + qu = f$$

on $(0, 1)$. Let v be another smooth function that also satisfies the boundary conditions. Multiply the ODE by v and integrate by parts to arrive at an expression of the form

$$a(u, v) = F(v)$$

where $a(u, v)$ involves integrals with u', v', u, v, k, b, and q, and $F(v)$ involves an integral with f and v. Now determine what Sobolev space V that u and v should belong to so that the bilinear form $a : V \times V \to \mathbb{R}$ and the linear form $F : V \to \mathbb{R}$ are well-defined, *and* which also incorporates the boundary conditions. The weak formulation is the following problem: find $u \in V$ such that

$$a(u, v) = F(v)$$

for all $v \in V$.

1.2 Problem 2

Such stability estimates are also called a priori (a Latin phrase meaning "from before") estimates. They are called such estimates because they are done *before* we actually know if we have a solution to the ODE. They always start in the following way: suppose that we have a solution $u \in V$ (where V is chosen in problem 1) such that

$$a(u, v) = F(v)$$

for all $v \in V$ (where a and F are also from problem 1). If you chose F correctly, you should be able to show that

$$F(v) \leq \|f\|_{L^2(0,1)} \|v\|_{L^2(0,1)}$$

for all $v \in V$. If you chose a correctly, you should be able to show that

$$a(u,u) \geq \overline{k}\|u'\|^2_{L^2(0,1)}$$

for all $u \in V$. The last ingredient you will need is the following Poincaré inequality:

Theorem 1. *Let $x_0 \in [a,b]$ and let $H^1_{x_0}(a,b)$ be the space of all functions $u \in H^1(a,b)$ such that $u(x_0) = 0$. Then there is a constant $C > 0$ such that*

$$\|u\|_{L^2(a,b)} \leq C\|u'\|_{L^2(a,b)}.$$

Proof. If u is a smooth function such that $u(x_0) = 0$, then for any $x > x_0$ we have that

$$u(x) = \int_{x_0}^{x} u'(t)\,\mathrm{d}t.$$

Therefore, by Cauchy-Schwarz,

$$|u(x)| \leq \int_{x_0}^{x} |u'(t)|\,\mathrm{d}t \leq \sqrt{b-a}\|u'\|_{L^2(a,b)}.$$

Now for $x < x_0$, we have that

$$u(x) = -\int_{x}^{x_0} u'(t)\,\mathrm{d}t,$$

so we can repeat a similar argument to conclude that

$$|u(x)| \leq \sqrt{b-a}\|u'\|_{L^2(a,b)}$$

for all $x \in [a,b]$. This implies that

$$\|u\|_{L^2(a,b)} \leq (b-a)\|u'\|_{L^2(a,b)}$$

for all smooth functions u such that $u(x_0) = 0$.

Now let $u \in H^1_{x_0}(a,b)$. Then since smooth functions that vanish at x_0 are dense in $H^1_{x_0}(a,b)$, there is a sequence $(u_n)_n$ of smooth functions that vanish at x_0 such that $\|u - u_n\|_{H^1(a,b)} \to 0$ as $n \to \infty$. Then $\|u - u_n\|_{L^2(a,b)} \to 0$ as $n \to \infty$ and $\|u_n'\|_{L^2(a,b)} \to \|u'\|_{L^2(a,b)}$ as $n \to \infty$. Then for each n,

$$\begin{aligned}\|u\|_{L^2(a,b)} &\leq \|u_n\|_{L^2(a,b)} + \|u - u_n\|_{L^2(a,b)} \\ &\leq (b-a)\|u_n'\|_{L^2(a,b)} + \|u - u_n\|_{L^2(a,b)} \to (b-a)\|u'\|_{L^2(a,b)}\end{aligned}$$

as $n \to \infty$. This finishes the proof. □

Combining everything together will give you the stability result.

2 Exercise 2

2.1 Problem 1

2.1.1 Part a

Multiply by a test function and integrate by parts. The boundary condition at $x = 1$ is something we have seen before, but now for the boundary condition at 0, use it to substitute for $u'(0)$. Rearrange everything and you will get something of the form

$$a(u, v) = F(v)$$

where $a(u, v)$ involves integrals with u', v' as well as values $u(0), v(0)$ and β, while $F(v)$ will involve an integral with f, v as well as the values $v(0), \gamma$, and β. Once again, look at the bilinear form a and the linear form F to decide which Sobolev space the functions u, v should belong to for the values $a(u, v)$ and $F(v)$ to be well-defined and to also incorporate the boundary conditions from the problem. Hint: you already included the boundary condition at 0 in a weak sense when you did the substitution, but now what about the boundary condition at $x = 1$?

2.1.2 Part b

Check the assumptions of the Lax-Milgram Theorem, which we recall below.

Theorem 2. *Let V be a Hilbert space with inner product $(\cdot, \cdot)_V$ and induced norm $\|v\|_V := \sqrt{(v, v)_V}$. Let $a : V \times V \to \mathbb{R}$ and $F : V \to \mathbb{R}$ be a bilinear form and a linear form on V respectively. Suppose that*

1. *a is continuous on V: there exists $C > 0$ such that*

$$|a(u, v)| \leq C\|u\|_V \|v\|_V$$

   for all $v \in V$

2. *F is continuous on V: there exists $C' > 0$ such that*

$$|F(v)| \leq C'\|v\|_V$$

   for all $v \in V$

3. *a is coercive (also known as elliptic) on V: there exists $\alpha > 0$ such that*

$$a(u, u) \geq \alpha\|u\|_V^2$$

   for all $u \in V$

Then there is a unique $u \in V$ such that

$$a(u, v) = F(v)$$

for all $v \in V$.

If you chose a, V, and F correctly in part a, you will be able to verify all of these assumptions. For the continuity assumptions, you will need the following, which is a corollary from some of the results in your last homework.

Theorem 3. *There is a constant C such that*

$$|u(x)| \leq C\|u\|_{H^1(a,b)}$$

for all $x \in [a,b]$ and all $u \in H^1(a,b)$.

For coercivity, you will need to use the Poincaré inequality that I showed earlier.

2.1.3 Part c

You can show either an estimate of the form

$$\|u\|_{H^1(0,1)} \leq E(f, \gamma, \beta)$$

or

$$\|u'\|_{L^2(0,1)} \leq \widetilde{E}(f, \gamma, \beta)$$

where u is the solution to the weak problem that we showed exists from part b and $E(f, \gamma, \beta)$ and $\widetilde{E}(f, \gamma, \beta)$ are some continuous expressions involving the function f and the boundary data γ and β. By the Poincaré inequality, we have that

$$\|u'\|_{L^2(0,1)} \leq \|u\|_{H^1(0,1)} \leq C\|u'\|_{L^2(0,1)}$$

so that the inequalities above are equivalent: one holds for some E iff the other holds for some $\widetilde{E}$. The argument is similar to stuff we have done earlier in the homework: you will have to use the coercivity of a, the continuity of F, and possibly the Poincaré inequality. Also, you cannot simply cite Lax-Milgram in this problem since it asks you to derive it yourself.

2.1.4 Part d

If $a(u_1, v) = F(v) = a(u_2, v)$ for all $v \in V$, then

$$a(u_1, v) - a(u_2, v) = 0$$

for all $v \in V$. Now use bilinearity and coercivity.

2.2 Problem 2

2.2.1 Part a

Suppose u and v are smooth, undo the integration by parts and use the boundary condition $u(1) = 0$ to get something of the form

$$\int_0^1 (Du - f)v \, \mathrm{d}x + (\text{boundary term at } x = 0) = 0$$

for all smooth v (and, by density, all $v \in V$), where Du is some expression involving u'', α, and u. Since V contains functions that vanish at $x = 0$, argue that this implies

$$\int_0^1 (Du - f)v \, \mathrm{d}x = 0 \text{ for all } v \in C_c^\infty(0,1)$$
$$\text{(boundary term at } x = 0) = 0 \text{ for all } v \in V$$

The hint in the homework tells you what ODE u satisfies on $(0,1)$, while picking v to be a smooth function that does not vanish at $x = 0$ in the boundary term equation will give you another boundary term that u must satisfy at $x = 0$. Therefore, your answer should be of the form

$$\begin{aligned} \text{ODE that } u \text{ satisfies } & \text{ on } (0,1) \\ \text{boundary condition at } & x = 0 \\ \text{boundary condition at } & x = 1 \end{aligned}$$

2.2.2 Part b

Same routine as the last energy estimates: use coercivity of the left side, continuity of the right side, and maybe a Poincaré inequality depending on if you're estimating $\|u\|_{H^1(0,1)}$ or $\|u'\|_{L^2(0,1)}$.