

MATH 610 Homework 4 Hints

Jordan Hoffart

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Problem 1

This problem is deceptively tricky. If we try to proceed as in the proof of Lemma 4.2.2 in the class notes, then we may be tempted to use the exact solution to the ODE

$$\begin{aligned} -W'' + bW' &= 1, & x \in (0, 1), \\ W(0) &= 0, \\ W(1) &= 0, \end{aligned}$$

which is

$$W(x) = \frac{1}{b(1 - e^b)}(e^{bx} - 1) + \frac{x}{b}.$$

We would then set

$$W_i = W(x_i)$$

for each i . Now, unlike Lemma 4.2.2, the W_i do *not* exactly satisfy the corresponding difference equation. That is, it is *not* true that the W_i satisfy

$$-\frac{W_{i+1} - 2W_i + W_{i-1}}{h^2} + b\frac{W_{i+1} - W_i}{h} = 1, \quad i \in \{1, \dots, N\} \quad (1a)$$

$$W_0 = 0, \quad (1b)$$

$$W_{N+1} = 0. \quad (1c)$$

One can see this by looking at Taylor series expansions of W around the points x_i using the points x_{i+1} and x_{i-1} . I leave the details to you, but the essential reason why the equation is not satisfied exactly is because W is not a polynomial. We didn't have this problem in Lemma 4.2.2 because the exact solution in that case *is* a polynomial, and so there is no remainder term in the Taylor series expansions.

We have to proceed in a slightly different way to tackle this problem. What we need is not the exact ODE solution W , but just a vector \vec{W} that satisfies the *system of difference equations* (1) exactly. Does such a vector exist? Well,

since the system (1) is a square system of linear equations, it suffices to show that if

$$-\frac{W_{i+1} - 2W_i + W_{i-1}}{h^2} + b\frac{W_{i+1} - W_i}{h} = 0, \quad i \in \{1, \dots, N\} \quad (2a)$$

$$W_0 = 0, \quad (2b)$$

$$W_{N+1} = 0, \quad (2c)$$

then $\vec{W} = \vec{0}$. We can show this by proving that this system has a discrete maximum principle and a discrete minimum principle. I will partially state these below, and I will leave it to you to prove these results as well as to come up with the condition on h that is sufficient for these lemmas to hold true.

Lemma 1 (Discrete maximum principle). *Suppose that h satisfies some stability condition that you must explicitly find. If \vec{W} satisfies*

$$-\frac{W_{i+1} - 2W_i + W_{i-1}}{h^2} + b\frac{W_{i+1} - W_i}{h} \leq 0, \quad i \in \{1, \dots, N\}$$

$$W_0 = 0,$$

$$W_{N+1} = 0,$$

then

$$\max_{i \in \{0, \dots, N+1\}} W_i = \max(W_0, W_{N+1}) = 0.$$

Corollary 1 (Discrete minimum principle). *Suppose that h satisfies the same stability condition from above. If \vec{W} satisfies*

$$-\frac{W_{i+1} - 2W_i + W_{i-1}}{h^2} + b\frac{W_{i+1} - W_i}{h} \geq 0, \quad i \in \{1, \dots, N\}$$

$$W_0 = 0,$$

$$W_{N+1} = 0,$$

then

$$\min_{i \in \{0, \dots, N+1\}} W_i = \min(W_0, W_{N+1}) = 0.$$

Corollary 2 (Uniqueness). *Under the stability condition on h , the square linear system (2) has only $\vec{W} = \vec{0}$ as its solution, and therefore there is a unique vector \vec{W}^0 that solves (1).*

Now that we have established the existence of \vec{W}^0 that satisfies (1), we can now proceed as in Lemma 4.2.2 from the class notes to finish the proof.

Problem 2.1

It's hard to give a hint for this that isn't essentially the entire proof. Instead, I will show a proof of the related result in 1d, but I will present the proof in a way that can be modified for higher dimensions. I will leave the details to you to provide this generalization.

Theorem 1. *There is a constant $C > 0$ such that*

$$\sqrt{v(0)^2 + v(1)^2} \leq C(\|v\|_{L^2(0,1)} + \|v'\|_{L^2(0,1)})$$

for all $v \in C^1[0, 1]$.

To prove this theorem, we will first establish a lemma.

Lemma 2. *If $v(0) = 0$, then*

$$|v(1)| \leq \|v'\|_{L^2(0,1)}.$$

Similarly, if $v(1) = 0$, then

$$|v(0)| \leq \|v'\|_{L^2(0,1)}.$$

Proof. For the first inequality, we use the fundamental theorem of calculus and write

$$v(1) = \int_0^1 v'(x) dx.$$

Then we take absolute values and apply Cauchy-Schwarz. The other inequality is proved in the exact same way. \square

Now we proceed with the proof of the theorem.

Proof. The trick here is to realize that we can write v as $v(x) = v_1(x) + v_2(x)$, where $v_1(x) = xv(x)$ vanishes at $x = 0$ and $v_2(x) = (1-x)v(x)$ vanishes at $x = 1$. Then we apply the previous lemma to v_i to get that

$$\begin{aligned} |v(1)| &= |v_1(1)| \leq \|v_1'\|_{L^2(0,1)} = \|v + xv'\|_{L^2(0,1)} \leq \|v\|_{L^2(0,1)} + \|v'\|_{L^2(0,1)}, \\ |v(0)| &= |v_2(0)| \leq \|v_2'\|_{L^2(0,1)} = \|(1-x)v' - v\|_{L^2(0,1)} \leq \|v\|_{L^2(0,1)} + \|v'\|_{L^2(0,1)}. \end{aligned}$$

Therefore,

$$\sqrt{v(0)^2 + v(1)^2} \leq |v(0)| + |v(1)| \leq 2(\|v\|_{L^2(0,1)} + \|v'\|_{L^2(0,1)}).$$

\square

As a hint for the generalization, we realize that we can write $v \in C^1(\widehat{K})$ as $v(\widehat{x}, \widehat{y}) = v_1(\widehat{x}, \widehat{y}) + v_2(\widehat{x}, \widehat{y}) + v_3(\widehat{x}, \widehat{y})$ where

$$\begin{aligned} v_1(\widehat{x}, \widehat{y}) &= \widehat{x}v(\widehat{x}, \widehat{y}) \text{ vanishes on the part of the boundary where } \widehat{x} = 0, \\ v_2(\widehat{x}, \widehat{y}) &= \widehat{y}v(\widehat{x}, \widehat{y}) \text{ vanishes on the part of the boundary where } \widehat{y} = 0, \\ v_3(\widehat{x}, \widehat{y}) &= (1 - \widehat{x} - \widehat{y})v(\widehat{x}, \widehat{y}) \text{ vanishes on the part of the boundary where } 1 - \widehat{x} - \widehat{y} = 0. \end{aligned}$$

Now just establish a version of Lemma 2 that bounds integrals of v_i over the boundary portion where $v_j = 0$ by some constant multiple of $\|\partial_x v_i\|_{L^2(\widehat{K})}$ or $\|\partial_y v_i\|_{L^2(\widehat{K})}$. There will be 6 inequalities in total of the form

$$\|v_i\|_{L^2(\widehat{E}_j)} \leq C\|\partial_k v_i\|_{L^2(\widehat{K})}$$

where \widehat{E}_j is one of the edges of \widehat{K} where v_i is *not* necessarily zero and ∂_k is one of the partial derivatives.

Problem 2.2

Denote the vertices as v_1 , v_2 , and v_3 . Consider the vectors formed from the differences $v_2 - v_1$ and $v_3 - v_1$. For the inequalities, recall that all norms on finite-dimensional vector spaces are equivalent. This can give you the result without knowing anything about the constants. Also, there is a typo in this question. To the best of my knowledge, the constants *will* depend on the diameter h of the triangle K . In fact, one can show that

$$\|DF_K\| := \sup_{\substack{\xi \in \mathbb{R}^2 \\ \xi \neq 0}} \frac{\|DF_K \xi\|}{\|\xi\|} \leq Ch$$

where C is a constant that does *not* depend on the diameter.

Problem 2.3

Time to get good at calculus. This is essentially an exercise in change-of-variables. First I will give you an unorganized collection of calculus facts that will help you do the analysis.

We recall that if f is a sufficiently smooth function on K , and if F_K is the map from \widehat{K} to K from the previous part, then the multivariable change-of-variables formula is

$$\int_K f(x) dx = \int_{\widehat{K}} f(F_K(\widehat{x})) |\det DF_K| d\widehat{x}.$$

Here, I am abusing notation and letting \widehat{x} and x denote the 2d coordinates on \widehat{K} and K respectively. Furthermore, if E_i is one of the edges of K , then we can enumerate the edges of \widehat{K} such that $F_K(\widehat{E}_i) = E_i$. Set $T_i = F_K|_{\widehat{E}_i}$, which maps \widehat{E}_i to E_i .

If you did the previous part correctly, you would know that the reference transformation F_K is an affine linear map of the form

$$F_K(\widehat{x}) = a + B\widehat{x}$$

for an invertible matrix B and a vector a , and that $DF_K = B$. Furthermore, from the change-of-variables formula above, we have that

$$|K| = \int_K 1 dx = \int_{\widehat{K}} |\det DF_K| d\widehat{x} = |\widehat{K}| |\det DF_K|,$$

where $|K|$ is the area of K and $|\widehat{K}| = 1/2$ is the area of \widehat{K} . Now since K is a triangle, its diameter h is its longest side. Since the area of a triangle is $1/2 \times \text{base} \times \text{height}$, if we set the base to h , then the height can be written as $C_K h$ where $0 < C_K < 1$ is a constant that depends on the particular triangle K but does not depend on its diameter h . Thus,

$$|\det DF_K| = 2|K| = C_K h^2.$$

Similarly, after parameterizing \widehat{E}_i and E_i as 1d intervals, T_i is of the form

$$T_i(\widehat{s}) = a + b\widehat{s}$$

for some scalars a and b , with $T'_i = b$. Then the change-of-variables formula on this edge would be

$$\int_{E_i} f(s) ds = \int_{\widehat{E}_i} f(T_i(\widehat{s}))|T'_i| d\widehat{s} = \int_{\widehat{E}_i} f(F_K|_{\widehat{E}_i}(\widehat{s}))|T'_i| d\widehat{s}.$$

Also, similar to above,

$$|E_i| = \int_{E_i} 1 ds = \int_{\widehat{E}_i} |T'_i| d\widehat{s} = |\widehat{E}_i||T'_i|.$$

Since the smallest edge length of \widehat{K} is 1 and the longest edge length of K is its diameter, we have that

$$|T'_i| = \frac{|E_i|}{|\widehat{E}_i|} \leq h.$$

Using this little grab-bag of calculus tricks, prove the result by first showing

$$\|v\|_{L^2(\partial K)} \leq h^{1/2} \|v \circ F_K\|_{L^2(\partial \widehat{K})}.$$

Then use part 1, part 2, and the calculus tricks above to finish it off. You will need the chain rule for the gradient:

$$\nabla(v \circ F_K) = (DF_K)^T((\nabla v) \circ F_K)$$

as well as

$$\|(DF_K)^T((\nabla v) \circ F_K)\|_{L^2(\widehat{K})} \leq \|DF_K\| \|(\nabla v) \circ F_K\|_{L^2(\widehat{K})}.$$