MATH 610 Homework 5 Hints

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Exercise 1

1. Multiply the first equation by a test function v and integrate by parts. Do the same for the second equation with a test function ψ. Then add them together.

2. Let $(\cdot, \cdot)_{\mathbb{V}}$ and $(\cdot, \cdot)_{\mathbb{W}}$ be inner products on $\mathbb{V}$ and $\mathbb{W}$ respectively. Recall that
$$\|v\|_{\mathbb{V}}^2 = (v, v)_{\mathbb{V}}$$
and
$$\|w\|_{\mathbb{W}}^2 = (w, w)_{\mathbb{W}}$$
are the norms on $\mathbb{V}$ and $\mathbb{W}$ respectively induced by these inner products. In the product space $\mathbb{V} \times \mathbb{W}$, we can define an inner product as
$$((u, \varphi), (v, \psi))_{\mathbb{V} \times \mathbb{W}} = (u, v)_{\mathbb{V}} + (\varphi, \psi)_{\mathbb{W}}.$$

Then the norm induced by this inner product is related to the norms on $\mathbb{V}$ and $\mathbb{W}$ via
$$\|(u, \varphi)\|_{\mathbb{V} \times \mathbb{W}}^2 = \|u\|_{\mathbb{V}}^2 + \|\varphi\|_{\mathbb{W}}^2.$$

This is the norm that we will use on $\mathbb{V} \times \mathbb{W}$ to do our analysis.

To show Lax-Milgram, you need to show that

(a) The bilinear form $a : (\mathbb{V} \times \mathbb{W}) \times (\mathbb{V} \times \mathbb{W}) \to \mathbb{R}$ is continuous, meaning that there is a constant $C > 0$ such that
$$|a((u, \varphi), (v, \psi))| \leq C \|(u, \varphi)\|_{\mathbb{V} \times \mathbb{W}} \|(v, \psi)\|_{\mathbb{V} \times \mathbb{W}}$$

for all $(u, \varphi), (v, \psi) \in \mathbb{V} \times \mathbb{W}$. You can show this directly, but it may be tedious. Another way that could save you some time would be to use the following fact.

Lemma 1. *Let X be a normed vector space with norm $\| \cdot \|$. If $a : X \times X \to \mathbb{R}$ and $b : X \times X \to \mathbb{R}$ are continuous bilinear forms on X, and if c is any scalar, then their linear combination $ca + b : X \times X \to \mathbb{R}$ defined by*
$$(ca + b)(x, y) = ca(x, y) + b(x, y)$$

is also a continuous bilinear form on X.

(b) The bilinear form a is coercive, meaning that there is a constant $\alpha > 0$ such that

$$a((u, \varphi), (u, \varphi)) \geq \alpha \|(u, \varphi)\|_{\mathbb{V} \times \mathbb{W}}^2$$

for all $(u, \varphi) \in \mathbb{V} \times \mathbb{W}$. If you chose your spaces $\mathbb{V}$ and $\mathbb{W}$ correctly, you can make use of a Poincaré inequality to help show this.

(c) The right-hand side defines a continuous linear functional on $\mathbb{V} \times \mathbb{W}$, meaning that if we set

$$F((v, \psi)) = \int_\Omega fv + g\psi \, dx,$$

then there is a constant $\Lambda > 0$ such that

$$|F((v, \psi))| \leq \Lambda \|(v, \psi)\|_{\mathbb{V} \times \mathbb{W}}$$

for all $(v, \psi) \in \mathbb{V} \times \mathbb{W}$. Once again, you can show this directly, or you can do something similar to above by using the following lemma:

Lemma 2. *Let X be a normed vector space with norm $\| \cdot \|$. If $a : X \to \mathbb{R}$ and $b : X \to \mathbb{R}$ are continuous linear forms on X, and if c is any scalar, then their linear combination $ca + b : X \to \mathbb{R}$ defined by*

$$(ca + b)(x) = ca(x) + b(x)$$

is also a continuous linear form on X.

The stability bound will involve the continuity of F and the coercivity of a.

3. First suppose that u is smooth and vanishes on the boundary. Use the fundamental theorem of calculus at a point $(x, y) \in \Omega_d$ and a point $(x, -d) \in \partial\Omega_d$. Then use Cauchy-Schwarz and then integrate x and y over Ω_d. If you bound things appropriately, you will get the result for smooth functions that vanish on the boundary. The full result follows from density of such functions in $H_0^1(\Omega_d)$. Just state the density result, but do not show it.

4. The only thing that should change should be the bilinear form and the fact that we work on Ω_d instead of Ω. The continuity of a and the continuity of the right-hand side should be more or less the same as the previous problem. What is different now is the coercivity argument. To show that, consider using the following trick: for any $u \in H_0^1(\Omega_d)$,

$$\|\nabla u\|_{L^2(\Omega_d)}^2 = \frac{1}{2}\|\nabla u\|_{L^2(\Omega_d)^2} + \frac{1}{2}\|\nabla u\|_{L^2(\Omega_d)^2}$$

$$\geq \frac{1}{2}\|\nabla u\|_{L^2(\Omega_d)^2} + \frac{1}{2c^2d^2}\|u\|_{L^2(\Omega_d)^2}$$

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and consider what happens when d is small enough such that

$$\frac{1}{2c^2d^2} > 1.$$

The stability bound will involve the continuity of F and the coercivity of a.

Exercise 2

1. There are a few different ways to approach this problem. We can either work on a general triangle K or the unit triangle $\widehat{K}$. The notes in Appendix C show that the general result follows from the unit triangle.

   Let us work on the reference triangle $\widehat{K}$ for concreteness. Since there is more than one way to affinely map the reference triangle onto K, let us agree for concreteness that $\widehat{a}_1 = (0,0)$, $\widehat{a}_2 = (1,0)$, $\widehat{a}_3 = (0,1)$, and the mapping is chosen such that vertex $\widehat{a}_i$ is mapped onto vertex a_i and midpoint $\widehat{a}_{ij}$ is mapped onto midpoint a_{ij}.

   Using the notation and definitions from Appendix C, the polynomial space $\widehat{P}$ that we work with on $\widehat{K}$ is still the set of $\mathbb{P}^2$ polynomials, but now on $\widehat{K}$. Similarly, the dofs $\widehat{\sigma}_i, \widehat{\sigma}_{ij} \in \widehat{\Sigma}$ on $\widehat{K}$ are essentially the same as the corresponding ones on K, just now we replace the vertices a_i on K by the vertices $\widehat{a}_i$ on $\widehat{K}$ and we replace the midpoints a_{ij} on K by the midoints $\widehat{a}_{ij}$ on $\widehat{K}$.

   The task now is to show that $\widehat{\Sigma}$ is a unisolvent set of dofs on $\widehat{K}$. From the equivalent (and more standard) definition in Appendix A, this means that we need to show if $\widehat{p} \in \widehat{P}$ satisfies $\widehat{\sigma}_i\widehat{p} = 0$ and $\widehat{\sigma}_{ij}\widehat{p} = 0$ for all i, j, then $\widehat{p} = 0$.

   Since we explicitly know the vertices and midpoints on $\widehat{K}$, and since we can write out a generic $\widehat{p} \in \widehat{P}$ as

   $$\widehat{p}(\widehat{x}, \widehat{y}) = a + b\widehat{x} + c\widehat{y} + d\widehat{x}\widehat{y} + e\widehat{x}^2 + f\widehat{y}^2,$$

   you can explicitly obtain a system of linear equations in the coefficients of the form

   $$A \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} = 0.$$

   Then if you show that A is invertible, you have unisolvence. While inverting a 6×6 matrix with a computer is trivial, and therefore good enough for the homework, for those that wish to do the qualifying exam, you do *not* want to waste your time inverting this matrix by hand. Therefore, I

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will show you another line of reasoning that is conceptually more difficult but is actually computationally feasible by hand.

The main idea that we will exploit comes from factoring multivariable polynomials. The relevant background information is included in Appendix D, from which the most important result is Lemma 10 as well as its corollaries. You do *not* need to know the proofs in that section, but they are included for completeness.

How we use the results from that section is as follows. We know that the degree of $\widehat{p}$ is at most 2. Using three points that lie on a straight line defined by some equation of the form $a\widehat{x} + b\widehat{y} + c = 0$ and which $\widehat{p}$ vanishes at the points, the factoring lemma tells us that, since $\widehat{p}$ vanishes at 3 points on the line, we can factor out the equation of the line from $\widehat{p}$. In other words, there is a degree at most 1 polynomial $\widehat{q}$ such that

$$\widehat{p}(\widehat{x}, \widehat{y}) = (a\widehat{x} + b\widehat{y} + c)\widehat{q}(\widehat{x}, \widehat{y}).$$

Now use two other points for which $\widehat{p}$ vanishes and which do *not* lie on the line $a\widehat{x} + b\widehat{y} + c = 0$ in order to factor $\widehat{q}$. If you did things correctly, you will be able to factor $\widehat{p}$ as

$$\widehat{p}(\widehat{x}, \widehat{y}) = (a\widehat{x} + b\widehat{y} + c)(d\widehat{x} + e\widehat{y} + f)C$$

for some constant C. Now pick a point that does not lie on the two lines and which $\widehat{p}$ vanishes and see what this implies.

2. Following the hints, we find the local shape functions $\widehat{\varphi}_i$ and $\widehat{\varphi}_{ij}$ on $\widehat{K}$ such that

$$\widehat{\sigma}_i\widehat{\varphi}_j = \delta_{ij}$$
$$\widehat{\sigma}_{ij}\widehat{\varphi}_k = 0$$
$$\widehat{\sigma}_k\widehat{\varphi}_{ij} = 0$$
$$\widehat{\sigma}_{ij}\widehat{\varphi}_{kl} = \delta_{ij,kl},$$

where the last equation means that $\widehat{\varphi}_{kl}$ takes the value 1 at dof $\widehat{\sigma}_{kl}$ and takes the value 0 at the other dofs. From the discussion in Appendices B and C and the previous part, there is exactly one set of functions that satisfies this, and they form a basis for $\widehat{P}$ the set of $\mathbb{P}^2$ polynomials on $\widehat{K}$.

The question now is how to actually find explicit formulas for these shape functions. Similar to the previous problem, there are two ways to do this, one more computationally feasible than the other.

The first approach, which is only practical if you have a computer to assist you, is to write out basis function with respect to the monomial basis $\{1, \widehat{x}, \widehat{y}, \widehat{x}\widehat{y}, \widehat{x}^2, \widehat{y}^2\}$. If we agree to relabel the dofs as $\widehat{\sigma}_4 = \widehat{\sigma}_{12}$, $\widehat{\sigma}_5 = \widehat{\sigma}_{23}$, and $\widehat{\sigma}_{31}$ as well as the corresponding shape functions, then we can write each shape function as

$$\widehat{\varphi}_i = a_i + b_i\widehat{x} + c_i\widehat{y} + d_i\widehat{x}\widehat{y} + e_i\widehat{x}^2 + f_i\widehat{y}^2$$

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for some coefficients that are to-be-determined. Then using the fact that

$$\widehat{\sigma}_j \widehat{\varphi}_i = \delta_{ij}$$

for all i, j (including the relabeling) gives us the following matrix system of the form

$$A \begin{pmatrix} a_1 & a_2 & \cdots & a_6 \\ b_1 & b_2 & \cdots & b_6 \\ \vdots & \vdots & \ddots & \vdots \\ f_1 & f_2 & \cdots & f_6 \end{pmatrix} = I,$$

where A is the coefficient matrix from the last problem and I is the 6×6 identity matrix. Now you just use a computer to assist you and invert A to get all the coefficients of the shape functions with respect to the monomial basis. Reading off the coefficients will allow you to write down explicitly what $\widehat{\varphi}_i$ is supposed to be.

The approach above is conceptually simple but practically inefficient and error prone, especially when you have to do this by hand on a qualifying exam. Here is the other approach, which is harder to understand but can be done by hand. It is similar to the previous problem, and will once again exploit some facts about factoring multivariable polynomials as in Appendix D.

Let us illustrate the idea for $\widehat{\varphi}_1$. The other shape functions can be found in a similar way. Reading off the equations from applying the dofs to $\widehat{\varphi}_1$ and simplifying tells us that

$$\widehat{\varphi}_1(\widehat{a}_1) = 1,$$
$$\widehat{\varphi}_1(\widehat{a}_2) = 0,$$
$$\widehat{\varphi}_1(\widehat{a}_3) = 0,$$
$$\widehat{\varphi}_1(\widehat{a}_{23}) = 0,$$
$$\widehat{\varphi}_1(\widehat{a}_{12}) = 1/2,$$
$$\widehat{\varphi}_1(\widehat{a}_{31}) = 1/2.$$

Now we use the factor theorem from Appendix D iteratively, first with the points $\widehat{a}_2, \widehat{a}_{23}$, and $\widehat{a}_3$. Then we use the (generalized) factor theorem again with the points $\widehat{a}_{12}$ and $\widehat{a}_{31}$. Finally, we use the point $\widehat{a}_1$. Doing everything correctly will give us an explicit formula for $\widehat{\varphi}_1$. Proceeding in a similar fashion also gives us the formulas for the other shape functions.

Proceeding with either approach gives you the shape functions on the reference element. Then, as explained in Appendix C, we obtain the physical shape functions φ_i by setting

$$\varphi_i = \widehat{\varphi}_i \circ T_K^{-1}.$$

Do not try to explicitly compute T_K^{-1} or φ_i. Just give me the basis functions $\widehat{\varphi}_i$ on $\widehat{K}$.

3. Just use a change-of-variables back to the reference triangle:

$$
\begin{aligned}
m_{11} &= \int_K \varphi_1(x)^2 \, dx \\
&= \int_{\widehat{K}} \varphi_1(T_K(\widehat{x}))^2 |\det DT_K| \, d\widehat{x} \\
&= 2|K| \int_{\widehat{K}} \widehat{\varphi}_1(\widehat{x})^2 \, d\widehat{x},
\end{aligned}
$$

where we used the fact (from a previous homework) that

$$
|K| = \int_K 1 \, dx = \int_{\widehat{K}} |\det DT_K| \, d\widehat{x} = \frac{1}{2} |\det DT_K|.
$$

Do not try to find the area of the general triangle K, but do use your work from the previous problem (and a computer to help you if you wish) to explicitly compute

$$
\int_{\widehat{K}} \widehat{\varphi}_1(\widehat{x})^2 \, d\widehat{x} = \int_0^1 \int_0^{\widehat{x}_2} \widehat{\varphi}_1(\widehat{x}_1, \widehat{x}_2)^2 \, d\widehat{x}_1 \, d\widehat{x}_2.
$$

Exercise 3

1. Take the general polynomial $p(x, y)$ and evaluate it at the dofs to get a system of equations involving the coefficients a, b, c, d where the right-hand side is set to 0. Show that this implies that $p = 0$.

2. Observe that all 4 midpoints have $1/2$ as one of the coordinates. Can you think of a degree 1 polynomial in one variable $q(t)$ that vanishes at $t = 1/2$? This will let you build a direct counterexample: a polynomial $p(x, y) \in \mathcal{P}$ such that $p \neq 0$ but p vanishes at all the dofs.

A Equivalent definitions of unisolvence

Let K be a triangle, P a space of polynomials on K, and Σ a set of dofs on P. We will show two definitiosn of unisolvence are equivalent. First, let's take the definition given in the homework, which is more intuitive.

Definition 1. Σ is unisolvent on P if any $p \in P$ is uniquely determined by its values on all $\sigma \in \Sigma$. In other words, Σ is unisolvent iff whenever $p, q \in P$ are such that $\sigma p = \sigma q$ for all $\sigma \in \Sigma$, we have that $p = q$.

Now we show that this is equivalent to the more standard definition, which is easier to check.

Lemma 3. Σ *is unisolvent iff whenever* $p \in P$ *satisfies* $\sigma p = 0$ *for all* $p \in P$, *we have that* $p = 0$.

Proof. For the forward direction, since the σ are linear, we just take $q = 0$ in the definition. For the reverse direction, if $p, q \in P$ are such that $\sigma p = \sigma q$ for all $\sigma \in \Sigma$, then once again by linearity we can set $r = p - q \in P$ and we have that $\sigma r = 0$ for all σ. This then implies that $r = 0$, so that $p = q$. This proves the other direction. $\square$

B Finite element triples and local shape functions

We recall the abstract definition of a finite element as a triple due to Ciarlet. We do not present the full definition, but only a special case.

Definition 2. Let K be a non-degenerate triangle in $\mathbb{R}^2$, P an n-dimensional space of polynomials on K, and Σ a set of n linear functionals $\sigma_1, \ldots, \sigma_n$ on P that we call degrees of freedom (dofs). The triple (K, P, Σ) is called a finite element if the map

$$p \in P \mapsto \Phi p = (\sigma_1 p, \ldots, \sigma_n p) \in \mathbb{R}^n$$

is a linear isomorphism.

Remark 1. More generally, K can be any domain in $\mathbb{R}^d$ for an arbitrary dimension d and P can be any n-dimensional space of functions on K (even vector-valued and not necessarily polynomial), but we do not need that for now.

The second condition says that, to know which polynomial we are working with, it suffices to know its values at the dofs. This is the essential idea of unisolvence. In other words, we have the following.

Lemma 4. *Let* (K, P, Σ) *be a triple as above. Then* (K, P, Σ) *is a finite element iff* Σ *is unisolvent.*

Proof. For the forward direction, if $\sigma_i p = 0$ for all i, then $\Phi p = 0$. Since Φ is an isomorphism, this implies $p = 0$, so that Σ is unisolvent.

Conversely, if Σ is unisolvent, then the linear map Φ is injective ($\Phi p = 0 \iff \sigma_i p = 0$ for all $i \implies p = 0$). However, since $\dim P = n = \dim \mathbb{R}^n$, the rank-nullity theorem from linear algebra implies that Φ is also surjective, i.e. it is a linear isomorphism. $\square$

The fact that Φ is a linear isomorphism (equivalently, that Σ is unisolvent) guarantees the existence and uniqueness of local shape functions with respect to the dofs. We now recall the definition of such shape functions.

Definition 3. Let (K, P, Σ) be a triple like above (not necessarily a finite element). Then a set of local shape functions on K with respect to this triple (if such a set exists) is a set of functions $\varphi_1, \ldots, \varphi_n \in P$ such that

$$\sigma_i \varphi_j = \delta_{ij}$$

for all i, j. We say that the φ_j are dual to the dofs if they have this property.

As promised, here is how unisolvence guarantees that such shape functions exist.

Lemma 5. *If (K, P, Σ) is a finite element triple (meaning that Σ is unisolvent), then there is exactly one set of local shape functions for this triple, and they form a basis for P.*

Proof. Let e_i be the ith standard basis vector of $\mathbb{R}^n$. Then we set $\varphi_i = \Phi^{-1} e_i \in P$, which is well-defined since Σ is unisolvent. Since $\Phi \varphi_i = e_i$ by construction, reading the coefficients tells us that

$$\sigma_j \varphi_i = \delta_{ij}$$

for all i, j. Thus the φ_i form a set of local shape functions that are dual to the σ_i. This shows existence.

Now we prove uniqueness. If ψ_i are another set of local shape functions, then by definition they satisfy

$$\sigma_i \psi_j = \delta_{ij}$$

for all i, j. However, this is equivalent to saying that

$$\Phi \psi_i = e_i$$

for each i. Since Φ is an isomorphism, this means that $\psi_i = \Phi^{-1} e_i = \varphi_i$ for all i. This proves uniqueness.

Finally, we show that the φ_i are a basis for P. Since each φ_i is distinct and there are $n = \dim P$ of them, it suffices to show that they are linearly independent. If

$$\sum_i c_i \varphi_i = 0$$

for some coefficients c_i, then by applying σ_j to both sides, we have that

$$c_j = \sum_i c_i \delta_{ij} = \sum_i c_i \sigma_j \varphi_i = \sigma_j\left(\sum_i c_i \varphi_i\right) = \sigma_j 0 = 0.$$

Thus $c_j = 0$ for all j, so that they are linearly independent and thus a basis for P. $\square$

Thus, unisolvence not only guarantees existence of shape functions, but also uniqueness and the fact that they form a basis.

C   Finite elements on the reference triangle

Let K be a non-degenerate triangle, let P be a space of polynomials on K, and let Σ be a set of degrees of freedom (dofs) on P, which is a set of linear functionals $\sigma : P \to \mathbb{R}$.

Now let $\widehat{K}$ be the reference triangle. Then we can map $\widehat{K}$ to K via an affine linear map $T_K : \widehat{K} \to K$. Furthermore, for any polynomial $p \in P$, $p \circ T_K$ is a polynomial on $\widehat{K}$ of the same degree.

Let

$$\widehat{P} = \{p \circ T_K : p \in P\}$$

be the collection of all such polynomials on $\widehat{K}$. Since K is a non-degenerate triangle, T_K is invertible, which implies that for any $\widehat{p} \in \widehat{P}$, there is a unique $p \in P$ such that

$$\widehat{p} = p \circ T_K.$$

In other words, the map $\psi_K : P \to \widehat{P}$ defined by $\psi_K p = p \circ T_K$ is a linear isomorphism. In fact, its inverse is just given by

$$\psi_K^{-1} \widehat{p} = \widehat{p} \circ T_K^{-1}.$$

Since ψ_K is a linear isomorphism, for any $\sigma \in \Sigma$, the composition $\sigma \circ \psi_K^{-1}$ is a dof on $\widehat{P}$. Let

$$\widehat{\Sigma} = \{\sigma \circ \psi_K^{-1} : \sigma \in \Sigma\}$$

be the collection of all such dofs on $\widehat{P}$. Similar to P and $\widehat{P}$, we have that, for every $\widehat{\sigma} \in \widehat{\Sigma}$, there is a unique $\sigma \in \Sigma$ such that

$$\widehat{\sigma} = \sigma \circ \psi_K^{-1},$$

namely,

$$\sigma = \widehat{\sigma} \circ \psi_K.$$

In other words, the association

$$\sigma \in \Sigma \mapsto \sigma \circ \psi_K^{-1} \in \widehat{\Sigma}$$

is a bijection between Σ and $\widehat{\Sigma}$ (in fact, it is also a linear isomorphism, but we don't need this).

This now gives us two different triples, the triple (K, P, Σ) on K and the triple $(\widehat{K}, \widehat{P}, \widehat{\Sigma})$ on $\widehat{K}$. With how we defined everything, we have the following relationship between them.

Lemma 6. Σ *is a unisolvent set of dofs on* P *iff* $\widehat{\Sigma}$ *is a unisolvent set of dofs on* $\widehat{P}$.

Proof. Suppose that Σ is unisolvent on P. From the previous section, this means that if $p \in P$ satisfies $\sigma p = 0$ for all $\sigma \in \Sigma$, then $p = 0$. Then if $\widehat{p} \in \widehat{P}$ is such that $\widehat{\sigma}\widehat{p} = 0$ for all $\widehat{\sigma} \in \widehat{\Sigma}$, we have that

$$(\widehat{\sigma} \circ \psi_K)(\psi_K^{-1}\widehat{p}) = 0$$

for all $\widehat{\sigma} \in \widehat{\Sigma}$. From our discussion above, this means that the polynomial $p = \psi_K^{-1}\widehat{p} \in P$ satisfies
$$\sigma p = 0$$

for all $\sigma \in \Sigma$. Since Σ is unisolvent, this implies that $p = 0$, which in turn implies that $\widehat{p} = \psi_K p = 0$. Thus $\widehat{\Sigma}$ is also unisolvent. This proves one direction. The other direction follows a similar argument. $\square$

Setting things up in this way gives us more than just a relation between unisolvence. It also gives us a relation between the local shape functions on the elements.

Suppose now that (K, P, Σ) is a finite element triple as in the previous section. This means that $\dim P = n$ and $\Sigma = \{\sigma_1, \ldots, \sigma_n\}$ is a unisolvent set of dofs. From how we set everything up, this also implies that (and is in fact equivalent to) $(\widehat{K}, \widehat{P}, \widehat{\Sigma})$ is a finite element triple, with $\dim \widehat{P} = n$ and $\widehat{\Sigma} = \{\widehat{\sigma}_1, \ldots, \widehat{\sigma}_n\}$ also being a unisolvent set of dofs.

Also from the previous section, we have a unique set of local shape functions $\varphi_1, \ldots, \varphi_n$ for the finite element on K as well as a unique set of local shape functions $\widehat{\varphi}_1, \ldots, \widehat{\varphi}_n$ on $\widehat{K}$. They are related in very much the same way that the polynomials in P and $\widehat{P}$ are related.

Lemma 7. *In the setting described above, the local shape functions are related via*
$$\varphi_i \circ T_K = \widehat{\varphi}_i$$

for each i. Equivalently,
$$\psi_K \varphi_i = \widehat{\varphi}_i.$$

Proof. By uniqueness of the shape functions, it suffices to show that

$$\widehat{\sigma}_i(\varphi_j \circ T_K) = \delta_{ij}$$

for all i, j. However, by unpacking our definitions, we have that $\sigma_i = \widehat{\sigma}_i \circ \psi_K$. Therefore,
$$\delta_{ij} = \sigma_i \varphi_j = \widehat{\sigma}_i(\psi_K \varphi_j) = \widehat{\sigma}_i(\varphi_j \circ T_K),$$

which is what we wanted to show. $\square$

D Factoring multivariable polynomials

We discuss when we can factor a multivariable polynomial. As a warmup, we recall some basic factoring results for 1d polynomials. The proofs in this section are included for completeness, and they utilize nothing more than Taylor's Theorem plus some inductive arguments.

Lemma 8. *Let p be a real-valued polynomial in 1 variable of degree $k \geq 1$. Then $p(a) = 0$ iff there is a polynomial q of degree $k - 1$ such that*

$$p(x) = (x - a)q(x)$$

for all $x \in \mathbb{R}$.

Proof. Fix x and set $h = a - x$. Performing a Taylor expansion of p at x using $x + h$ and using the fact that $p(a) = p(x + h) = 0$ tells us that

$$0 = p(x) + (a - x)p'(x) + \cdots + \frac{(a - x)^k}{k!}p^{(k)}(x).$$

Thus

$$p(x) = (x - a)p'(x) + \cdots + (-1)^{k+1}\frac{(x - a)^k}{k!}p^{(k)}(x).$$

Setting

$$q(x) = p'(x) - \frac{(x - a)}{2}p''(x) + \cdots + (-1)^{k+1}\frac{(x - a)^{k-1}}{k!}p^{(k)}(x)$$

proves one direction. The other direction holds from evaluating at $x = a$. $\square$

Corollary 1. *If p is a degree at most n polynomial that vanishes at $n+1$ points, then $p = 0$.*

Proof. Suppose p vanishes at the points $a_0, \ldots, a_n$. Then since p vanishes at a_n, we have that $p(x) = (x - a_n)q(x)$ where the degree of q is at most $n-1$. Since a_n is distinct from the other a_i but p vanishes at the other a_i, we must have that q is a degree at most $n-1$ polynomial that vanishes at n distinct points. Repeating this argument inductively allows us to conclude that $p(x) = (x - a_n) \cdots (x - a_1)C$ for some constant C. But $p(a_0) = 0$ and a_0 is distinct from the other a_i, so we must have that $C = 0$. Thus $p = 0$ identically. $\square$

This has a very easy extension to polynomials of more than one variable, which we now show.

Corollary 2. *Let p be a real-valued polynomial in m variables of total degree at most n. If p vanishes at $n+1$ points lying along a straight line, then p vanishes on that line.*

Proof. Parameterize the line with a degree one vector-valued polynomial $r(t)$. Then $p(r(t))$ is a degree at most n polynomial that vanishes at $n + 1$ distinct points $t_0, \ldots, t_n$. From the previous corollary, $p(r(t)) = 0$ for all t, which means that $p = 0$ on the line. $\square$

Now we generalize the factoring lemma for 1d polynomials to 2d polynomials. This could be generalized further to polynomials of any finite number of variables, but we will keep it simple and only discuss the 2d case. We first recall the 2d version of Taylor's Theorem, which follows from the 1d version plus some chain rule.

Lemma 9. *Let p be a polynomial in 2 variables of total degree n. Then for $(x, y) \in \mathbb{R}^2$,*

$$p(x + h, y + k) = p(x, y) + \sum_{m=1}^{n} \frac{1}{m!} \sum_{i=0}^{m} \binom{m}{i} \partial_x^{m-i} \partial_y^i p(x, y) h^{m-i} k^i.$$

Proof. Let $r(t) = (x + ht, y + kt)$. Then let $q(t) = p(r(t))$. Then q is a polynomial of degree n in t, so by Taylor's Theorem,

$$q(1) = q(0) + q'(0) + \cdots + \frac{1}{n!} q^{(n)}(0).$$

By the chain rule, we have that

$$q'(0) = \partial_x p(x, y) h + \partial_y p(x, y) k,$$
$$q''(0) = \partial_x^2 p(x, y) h^2 + 2 \partial_{xy}^2 p(x, y) hk + \partial_y^2 p(x, y) k^2,$$
$$\vdots$$
$$q^{(n)}(0) = \sum_{i=0}^{n} \binom{n}{i} \partial_x^{n-i} \partial_y^i p(x, y) h^{n-i} k^i.$$

Putting this altogether gives us the result. $\qquad\square$

Here is now the main result of this section, which gives us a condition for when we can factor a degree one polynomial term out of a multivariable polynomial.

Lemma 10. *Let p be a real-valued polynomial in 2 variables of total degree $n \geq 1$. Let L be the line consisting of all points (x, y) such that $ax + by + c = 0$. Then p vanishes on L iff there is a polynomial q of total degree $n - 1$ such that*

$$p(x, y) = (ax + by + c) q(x, y)$$

for all $(x, y) \in \mathbb{R}^2$.

Proof. Suppose without loss of generality that $a, b, c, \neq 0$, as these special cases are easier and handled similarly. Fix $(x, y) \in \mathbb{R}^2$ and consider the point $(x, -(ax + c)/b)$ on L with the same x coordinate. Then by applying the previous lemma with $h = 0$ and $k = -(ax + c)/b - y$, we have that

$$0 = p(x, y) + \sum_{m=1}^{n} \frac{1}{m!} \partial_y^m p(x, y) (-1)^m ((ax + c)/b + y)^m.$$

Therefore, after rearranging and factoring a term,

$$p(x,y) = (ax + by + c)\frac{1}{b}\sum_{m=1}^{n}\frac{(-1)^{m+1}}{m!}\partial_y^m p(x,y)(y + (ax+c)/b)^{m-1}.$$

Setting

$$q(x,y) = \frac{1}{b}\sum_{m=1}^{n}\frac{(-1)^{m+1}}{m!}\partial_y^m p(x,y)(y + (ax+c)/b)^{m-1}$$

proves one direction. The other direction holds by evaluating at a point on L. $\square$

To see if a degree n polynomial vanishes on a line, it suffices to know that it vanishes at $n+1$ points that lie on that line.

Corollary 3. *If a degree $n \geq 1$ polynomial p in 2 variables vanishes at $n+1$ points that lie on a straight line, and if the line is characterized as the set of solutions to $L(x,y) = 0$ for a degree one polynomial L, then $p(x,y) = L(x,y)q(x,y)$ for some degree $n-1$ polynomial q.*

Finally, the value 0 is nothing special. We obtain a slightly more general result if we know that a polynomial is constant on a line, or, equivalently, if a polynomial takes the same value at $n+1$ points that lie on a straight line.

Corollary 4. *If a degree $n \geq 1$ polynomial in 2 variables takes the same value C at $n+1$ points that lie on a straight line, and if the line is characharacterized as the set of solutions to $L(x,y) = 0$ for a degree one polynomial L, then $p(x,y) = L(x,y)q(x,y) + C$ for some degree $n-1$ polynomial q.*

Proof. Apply the previous corollary to the polynomial $r(x,y) = p(x,y) - C$. $\square$