MATH 610 Homework 6 Hints

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1. Show unisolvence. That is, suppose that $p \in P$ is such that $\sigma_i p = 0$ for all i. Since p is a piecewise quadratic, if we label $K_1 = [0, 1/2]$ and $K_2 = [1/2, 1]$, then there are quadratic polynomials p_1, p_2 such that $p|_{K_i} = p_i$. The dofs will allow you to factor the p_i, and the continuity conditions at $x = 1/2$ give you two more equations

$$p_1(1/2) = p_2(1/2),$$
$$p_1'(1/2) = p_2'(1/2).$$

You will want to use the following factoring results throughout the problem.

Lemma 1. *Let q be a quadratic polynomial.*

(a) If $q(0) = 0$, then $q(x) = x(ax + b)$ for some constants a, b.
(b) If $q'(0) = 0$, then $q(x) = ax^2 + b$ for some constants a, b.
(c) If $q(0) = q'(0) = 0$, then $q(x) = ax^2$ for some constant a.
(d) If $q(1) = 0$, then $q(x) = (x - 1)(ax + b)$ for some constants a, b.
(e) If $q'(1) = 0$, then $q(x) = ax^2 - 2ax + b$ for some constants a, b.
(f) If $q(1) = q'(1) = 0$, then $q(x) = a(x - 1)^2$ for some constant a.

Proof. Results a and d are just the usual factoring lemma.

Since q is a quadratic polynomial, q' is a degree one polynomial. Then from the usual factoring lemma, assumption b implies that $q'(x) = cx$ for some constant c. This in turn implies $q(x) = ax^2 + b$ for some constants a, b (namely, $a = c/2$).

Similarly, for assumption e, we have that $q'(x) = c(x-1)$ for some constant c. This implies that $q(x) = (c/2)x^2 - cx + b$ for a constant b. Setting $a = c/2$ gives us $q(x) = ax^2 - 2ax + b$.

For item c, we use item b and evaluate at $x = 0$ to conclude that $b = 0$. For item f, we have from item d that $q(x) = (x-1)(ax+b)$ for some constants a and b. Then by taking a derivative, we have that $q'(x) = ax+b+a(x-1)$. Evaluating at $x = 1$ tells us that $a + b = 0$, so that $q(x) = a(x - 1)^2$ as desired. □

To find the shape functions, we have that shape function $\varphi_i \in P$ satisfies $\sigma_j \varphi_i = \delta_{ij}$ for all i, j. We also have that $\varphi_i|_{K_k} = \varphi_{i,k}$ for some quadratic polynomials $\varphi_{i,k}$. Use the lemma above and the equations $\sigma_j \varphi_i = 0$ to factor the $\varphi_{i,k}$ as much as possible. Then use the equations

$$\sigma_i \varphi_i = 1$$
$$\varphi_{i,1}(1/2) = \varphi_{i,2}(1/2)$$
$$\varphi'_{i,1}(1/2) = \varphi'_{i,2}(1/2)$$

to solve for the coefficients that appear from factoring.

2. (a) Use a Poincaré inequality.

   (b) Lax-Milgram.

   (c) First show Galerkin orthgonality:

   $$a_k(u - u_h, v_h) = 0$$

   for all $v_h \in \mathbb{V}_h$. Then use coercivity, Galerkin orthogonality, and continuity to show Ceá's lemma: there is a constant C such that

   $$\|u - u_h\|_1 \leq C \inf_{v_h \in \mathbb{V}_h} \|u - v_h\|_1$$

   for all h. Then use Ceá's lemma and the given approximation property to bound $\|u - u_h\|_1^2$ above.

   (d) Let $g = u - u_h = v$. Then

   $$\|u - u_h\|^2 = (g, v) = a_k(w, v).$$

   Now use the fact that a_k is symmetric, use Galerkin orthogonality, and use continuity to show that

   $$a_k(w, v) \leq C\|u - u_h\|_1 \inf_{w_h \in \mathbb{V}_h} \|w - w_h\|_1.$$

   Combine these, use Ceá's lemma, apply the approximation result from above, and use the regularity assumption to finish the proof.