MATH 610 Homework 7 Hints

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Exercise 1

1. This is a standard problem that we have seen before, and this is also a standard Lax-Milgram argument. Determine V, $a : V \times V \to \mathbb{R}$ and $L : V \to \mathbb{R}$ such that the weak formulation is to find $u \in V$ such that

   $$a(u, v) = L(v)$$

   for all $v \in V$. Choose an appropriate norm $\|\cdot\|_V$ on V, and then show that a is continuous and coercive and L is continuous on V.

   For the purposes of a later problem, we make a few remarks about the constant of continuity and the constant of coercivity for a. The constant of continuity for a will depend on q and we denote it by C_q. That is,

   $$|a(u, v)| \leq C_q \|u\|_V \|v\|_V$$

   for all $u, v \in V$. Moreover, if you do things correctly, you can show that there is a constant $C > 0$ independent of q such that $C_q \to C$ as $q \to 0$.

   The constant of coercivity may also depend on q, and we denote it by β_q. That is,

   $$a(u, u) \geq \beta_q \|u\|_V^2$$

   for all $u \in V$. If you choose V correctly, then you can use a Poincaré inequality to show that the coercivity constant can be chosen independently of q. That is, you can find $\beta > 0$ independent of q such that

   $$a(u, u) \geq \beta \|u\|_V^2$$

   for all $u \in V$.

2. Observe that we are doing a conforming approximation and that continuity and coercivity are preserved on subspaces.

3. This is similar to what we did on a previous homework. Show that Galerkin orthogonality holds. Then show that Ceá's Lemma holds: there is a constant $C'_q > 0$ such that

   $$\|u - u_h\|_{H^1(\Omega)} \leq C'_q \inf_{v_h \in V_h} \|u - v_h\|_{H^1(\Omega)}.$$

You can use without proof the following approximation property for V_h, which is also a hint from a previous homework: there is a constant $C > 0$ such that

$$\inf_{v_h \in V_h} \|v - v_h\|_{H^1(\Omega)} \leq Ch\|v\|_{H^2(\Omega)}$$

for all $v \in H^2(\Omega)$. Combining these results will show that there is a constant $c_{1,q} > 0$ such that

$$\|u - u_h\|_{H^1(\Omega)} \leq c_{1,q}h\|u\|_{H^2(\Omega)}$$

for all $h > 0$.

The constant $c_{1,q}$ will in general depend on q, but if you do things correctly, then you can show that there is a constant $c_1 > 0$ such that $c_{1,q} \to c_1$ as $q \to 0$. This will be needed in a later problem.

4. Show that, since $u \in H^2(\Omega)$, then

$$-\Delta u + qu = f.$$

You can use without proof that if

$$\int_\Omega w\varphi = 0$$

for all $\varphi \in C_0^\infty(\Omega)$, then $w = 0$. Then, by working with components, show that the integration by parts lemma for scalar-valued H^1 functions implies that, for a vector-valued function v with each component $v_i \in H^1(\Omega)$ and a scalar-valued $w \in H^1(\Omega)$, we have the following version of integration-by-parts:

$$\int_\Omega \nabla \cdot vw = -\int_\Omega v \cdot \nabla w + \int_{\partial\Omega} n \cdot vw$$

Combine these to get the formula for α.

Now, to get the error estimate for $\alpha - \alpha_h$, use continuity and Galerkin orthogonality to show

$$|\alpha - \alpha_h| \leq C_q'\|u - u_h\|_{H^1(\Omega)} \inf_{w_h \in V_h} \|w - w_h\|_{H^1(\Omega)}.$$

Then use the previous part and the given approximation property of V_h to get

$$|\alpha - \alpha_h| \leq c_{2,q}h^2\|u\|_{H^2(\Omega)}.$$

Here, the constant $c_{2,q}$ will depend on q but not on h or u.

Now, for the case $q = 0$, walk back through your arguments for the previous questions and modify them for the $q = 0$ case. If you do things correctly, you will be able to show that, with small modifications, all of the arguments will carry through, just now with new constants.

Exercise 2

1. First, since $w_h|_K$ is affine-linear on K, then ∇w_h is constant on K. Furthermore, for an edge e contained in K on the boundary, the outward normal n is constant on e. Therefore, we have the following tricks:

$$\left(\int_e |n \cdot \nabla w_h|^2\right)^{1/2} = \ell_e^{1/2} |n|_e \cdot \nabla w_h|_K|,$$

$$\left(\int_K |\nabla w_h|^2\right)^{1/2} = |K|^{1/2} |\nabla w_h|_K|,$$

where ℓ_e is the length of the edge e and $|K|$ is the area of the triangle. Using these tricks and Cauchy-Schwarz, show that

$$\int_e n \cdot \nabla w_h v_h \leq \frac{\ell_e^{1/2}}{|K|^{1/2}} \left(\int_K |\nabla w_h|^2\right)^{1/2} \left(\int_e v_h^2\right)^{1/2}.$$

Now we derive a few useful facts from shape-regularity and quasi-uniformity. Recall that shape-regularity means that there is a constant $C > 0$ such that

$$\frac{h_K}{\rho_K} \leq C$$

for all triangles $K \in \mathcal{T}_h$ and all $h > 0$. Here, h_K is the diameter of the triangle and ρ_K is the diameter of the largest circle that can fit inside the triangle. Using the area of the circle, this implies that

$$|K| \geq \frac{1}{2}\pi(\rho_K/2)^2 \geq C h_K^2$$

for some constant C independent of h and K. Using this, show that

$$\frac{\ell_e}{|K|} \leq \frac{C}{h_K}$$

for some constant C independent of h and K.

Now we recall that quasi-uniformity means that there is a constant $C > 0$ such that

$$\frac{h}{h_K} \leq C$$

for all $K \in \mathcal{T}_h$ and all $h > 0$. Combine all of our observations to get the final estimate.

2. First, we define a norm on V_h. Let

$$\|u_h\|_h = \left(\|\nabla u_h\|_{L^2(\Omega)}^2 + \frac{\alpha}{h}\|u_h\|_{L^2(\partial\Omega)}^2\right)^{1/2}$$

for all $u_h \in V_h$. Show that this is a norm on V_h. You do not need to prove the triangle inequality or the homogeneity property. I only want to see you show that if $\|u_h\|_h = 0$, then $u_h = 0$.

Now, with this norm, show that a_h is continuous and coercive and L is continuous on V_h, with a_h and L being the left-hand and right-hand sides of the given equation we are seeking a solution for. Recall, or accept without proof, that for a finite-dimensional vector space V_h, any bilinear form or linear form is automatically continuous on V_h. Therefore, I do not want you to show continuity. I only want you to show that a_h is coercive on V_h with the given norm. For that, observe that

$$\begin{aligned} a_h(u_h, u_h) &= \|u_h\|_h^2 - \int_{\partial\Omega} n \cdot \nabla u_h u_h \\ &= \frac{1}{2}\|u_h\|_h^2 + \underbrace{\frac{1}{2}\|u_h\|_h^2 - \int_{\partial\Omega} n \cdot \nabla u_h u_h}_{(*)} . \end{aligned}$$

Now, let $\mathcal{T}_h^\partial$ be the set of all mesh cells that have an edge on the boundary, and let $\mathcal{T}_h^\circ = \mathcal{T}_h \setminus \mathcal{T}_h^\partial$. For each $K \in \mathcal{T}_h^\partial$, let $\mathcal{E}_K^\partial$ be the set of edges of K that lie on the boundary. Observe that we can write

$$\int_\Omega |\nabla u_h|^2 = \sum_{K \in \mathcal{T}_h^\circ} \int_K |\nabla u_h|^2 + \sum_{K \in \mathcal{T}_h^\partial} \int_K |\nabla u_h|^2$$

and

$$\int_{\partial\Omega} \frac{\alpha}{h} u_h^2 - n \cdot \nabla u_h u_h = \sum_{K \in \mathcal{T}_h^\partial} \sum_{e \in \mathcal{E}_K^\partial} \int_e \frac{\alpha}{h} u_h^2 - n \cdot \nabla u_h u_h.$$

Use these observations to start bounding $a_h(u_h, u_h)$ from below in a way that allows you to apply the previous estimate. If you do things correctly, you can then apply Young's inequality

$$a^2 + b^2 \geq 2ab$$

with

$$a = \left(\int_K |\nabla u_h|^2 \right)^{1/2}$$

and

$$b = \left(\frac{\alpha}{h} \int_e u_h^2 \right)^{1/2} .$$

If you do this correctly, then you can conclude that for $\alpha \geq C > 0$ with some constant C independent of h, we have that the term $(*)$ above is non-negative, which then implies coercivity. Conclude with Lax-Milgram.