Calculus

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1 Line integrals

1.1 Scalar fields

The line integral of a scalar-valued function over a curve C in $\mathbb{R}^n$ is calculated as

$$\int_C f\,ds = \int_a^b f(r(t))|r'(t)|\,dt$$

where r is a parameterization of the curve. The line integral is independent of choice of parameterization.

1.2 Vector fields

If F is a vector field, then the line integral along C is computed as

$$\int_C F \cdot dr = \int_a^b F(r(t)) \cdot r'(t)\,dt$$

where, once again, r is a parameterization of the curve. Unlike the scalar field case, the integral does depend on the choice of parameterization. However, it only depends on the orientation. That is, any other parameterization gives the same absolute value, but possibly a different sign.

1.2.1 Path independence

Fix points x_0, x_1 in $\mathbb{R}^n$. Let C be any curve from x_0 to x_1. Let r be a parameterization of C with $r(a) = x_0$ and $r(b) = x_1$. If F is a conservative vector field, meaning that there is a scalar field f such that $F = \nabla f$, then we have that

$$\begin{aligned} f(x_1) - f(x_0) &= f(r(b)) - f(r(a)) \\ &= \int_a^b \frac{d}{dt}(f \circ r)(t)\,dt \\ &= \int_a^b F(r(t)) \cdot r'(t)\,dt \\ &= \int_C F \cdot dr. \end{aligned}$$

Therefore, the line integral of a conservative vector field along a curve C depends only on its endpoints. It does not depend on the particular choice of curve.

2 Change of variables

Let R, S be two regions in $\mathbb{R}^n$ and let $\varphi : R \to S$ be a diffeomorphism. Then for any function f on S, we have the following change-of-coordinates formula

$$\int_S f = \int_R (f \circ \varphi)|\det D\varphi|,$$

where $D\varphi$ is the Jacobian of φ.

3 Chain rule

If $f : \mathbb{R}^n \to \mathbb{R}^m$ and $g : \mathbb{R}^m \to \mathbb{R}^p$, then $g \circ f : \mathbb{R}^m \to \mathbb{R}^p$. Looking at the ith coordinate function and taking the jth partial derivative allows us to apply the chain rule to get

$$\partial_j (g_i \circ f) = \sum_k ((\partial_k g_i) \circ f) \partial_j f_k.$$

If we denote the Jacobian D as the matrix with (i, j) entry the jth partial derivative of the ith coordinate function, then we can write the above equation as

$$D(g \circ f)_{i,j} = \sum_k ((Dg)_{i,k} \circ f)(Df)_{k,j}.$$

In other words,

$$D(g \circ f) = ((Dg) \circ f)Df.$$

Pointwise, and with a pedantic amount of parentheses,

$$(D(g \circ f))(x) = ((Dg)(f(x)))((Df)(x)).$$

3.1 Change of variables for a derivative

If the Jacobian of f is invertible as a matrix, then we have that

$$(Dg) \circ f = D(g \circ f)\ (Df)^{-1}.$$

If we then combine this with our change-of-variables result from above, we have that

$$\int_S Dg = \int_R ((Dg) \circ f)|\det Df| = \int_R (D(g \circ f)\ (Df)^{-1})|\det Df|$$

which is the change-of-variables formula for a derivative Dg with coordinate transformation $f : R \to S$.

3.2 The derivative of an inverse

Suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism. Then by the chain rule above, we have that

$$I = D(f \circ f^{-1}) = ((Df) \circ (f^{-1}))\ D(f^{-1}).$$

This implies that

$$D(f^{-1}) = (Df)^{-1} \circ (f^{-1}).$$

That is, the Jacobian of the inverse map is just the matrix inverse of the Jacobian (composed with the inverse). In particular, if Df is a constant matrix, then so is $D(f^{-1})$, and we have that

$$D(f^{-1}) = (Df)^{-1}.$$

That is, in this special case, the Jacobian of the inverse map is the matrix inverse of the Jacobian.

4 Divergence theorem

The divergence of a vector field F is defined as

$$\nabla \cdot F = \sum_i \partial_i F_i.$$

We recall the divergence theorem, which states that

$$\int_D \nabla \cdot F \, dx = \int_{\partial D} F \cdot n \, ds.$$

4.1 Integration by parts in higher dimensions

We now derive an integration-by-parts formula for higher dimensions. We first observe the following product rule for the divergence operator.

Lemma 1. *For a vector field F and a scalar field g,*

$$\nabla \cdot (Fg) = (\nabla \cdot F)g + F \cdot \nabla g.$$

Proof. We just unpack the definitions and use the product rule from 1d.

$$\begin{aligned}
\nabla \cdot (Fg) &= \sum_i \partial_i (F_i g) \\
&= \sum_i \partial_i F_i g + F_i \partial_i g \\
&= (\nabla \cdot F)g + F \cdot \nabla g.
\end{aligned}$$

□

Combining this with the divergence theorem gives us an integration-by-parts formula.

Lemma 2. *For a vector field F and a scalar field g,*

$$\int_D (\nabla \cdot F) g \, dx = \int_{\partial D} Fg \cdot n \, ds - \int_D F \cdot \nabla g \, dx.$$

Observe that, in dimension 1, the divergence and gradient operators reduce to the standard derivative operator. If D is then an interval, the outward normal n is $+1$ at the right endpoint and -1 at the left endpoint. The equation above then reduces to

$$\int_a^b F' g \, dx = Fg|_a^b - \int_a^b F g' \, dx$$

which is indeed the usual integration-by-parts formula in 1d.