Some remarks about finite element triples

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1 Equivalent definitions of unisolvence

Let K be a triangle, P a space of polynomials on K, and Σ a set of dofs on P. We will show two definitiosn of unisolvence are equivalent. First, let's take the definition given in the homework, which is more intuitive.

Definition 1. Σ is unisolvent on P if any $p \in P$ is uniquely determined by its values on all $\sigma \in \Sigma$. In other words, Σ is unisolvent iff whenever $p, q \in P$ are such that $\sigma p = \sigma q$ for all $\sigma \in \Sigma$, we have that $p = q$.

Now we show that this is equivalent to the more standard definition, which is easier to check.

Lemma 1. *Σ is unisolvent iff whenever $p \in P$ satisfies $\sigma p = 0$ for all $p \in P$, we have that $p = 0$.*

Proof. For the forward direction, since the σ are linear, we just take $q = 0$ in the definition. For the reverse direction, if $p, q \in P$ are such that $\sigma p = \sigma q$ for all $\sigma \in \Sigma$, then once again by linearity we can set $r = p - q \in P$ and we have that $\sigma r = 0$ for all σ. This then implies that $r = 0$, so that $p = q$. This proves the other direction. $\square$

2 Finite element triples and local shape functions

We recall the abstract definition of a finite element as a triple due to Ciarlet. We do not present the full definition, but only a special case.

Definition 2. Let K be a non-degenerate triangle in $\mathbb{R}^2$, P an n-dimensional space of polynomials on K, and Σ a set of n linear functionals $\sigma_1, \ldots, \sigma_n$ on P that we call degrees of freedom (dofs). The triple (K, P, Σ) is called a finite element if the map

$$p \in P \mapsto \Phi p = (\sigma_1 p, \ldots, \sigma_n p) \in \mathbb{R}^n$$

is a linear isomorphism.

Remark 1. More generally, K can be any domain in $\mathbb{R}^d$ for an arbitrary dimension d and P can be any n-dimensional space of functions on K (even vector-valued and not necessarily polynomial), but we do not need that for now.

The second condition says that, to know which polynomial we are working with, it suffices to know its values at the dofs. This is the essential idea of unisolvence. In other words, we have the following.

Lemma 2. *Let (K, P, Σ) be a triple as above. Then (K, P, Σ) is a finite element iff Σ is unisolvent.*

Proof. For the forward direction, if $\sigma_i p = 0$ for all i, then $\Phi p = 0$. Since Φ is an isomorphism, this implies $p = 0$, so that Σ is unisolvent.

Conversely, if Σ is unisolvent, then the linear map Φ is injective ($\Phi p = 0 \iff \sigma_i p = 0$ for all $i \implies p = 0$). However, since $\dim P = n = \dim \mathbb{R}^n$, the rank-nullity theorem from linear algebra implies that Φ is also surjective, i.e. it is a linear isomorphism. $\square$

The fact that Φ is a linear isomorphism (equivalently, that Σ is unisolvent) guarantees the existence and uniqueness of local shape functions with respect to the dofs. We now recall the definition of such shape functions.

Definition 3. Let (K, P, Σ) be a triple like above (not necessarily a finite element). Then a set of local shape functions on K with respect to this triple (if such a set exists) is a set of functions $\varphi_1, \dots, \varphi_n \in P$ such that

$$\sigma_i \varphi_j = \delta_{ij}$$

for all i, j. We say that the φ_j are dual to the dofs if they have this property.

As promised, here is how unisolvence guarantees that such shape functions exist.

Lemma 3. *If (K, P, Σ) is a finite element triple (meaning that Σ is unisolvent), then there is exactly one set of local shape functions for this triple, and they form a basis for P.*

Proof. Let e_i be the ith standard basis vector of $\mathbb{R}^n$. Then we set $\varphi_i = \Phi^{-1} e_i \in P$, which is well-defined since Σ is unisolvent. Since $\Phi \varphi_i = e_i$ by construction, reading the coefficients tells us that

$$\sigma_j \varphi_i = \delta_{ij}$$

for all i, j. Thus the φ_i form a set of local shape functions that are dual to the σ_i. This shows existence.

Now we prove uniqueness. If ψ_i are another set of local shape functions, then by definition they satisfy

$$\sigma_i \psi_j = \delta_{ij}$$

for all i, j. However, this is equivalent to saying that

$$\Phi\psi_i = e_i$$

for each i. Since Φ is an isomorphism, this means that $\psi_i = \Phi^{-1}e_i = \varphi_i$ for all i. This proves uniqueness.

Finally, we show that the φ_i are a basis for P. Since each φ_i is distinct and there are $n = \dim P$ of them, it suffices to show that they are linearly independent. If

$$\sum_i c_i\varphi_i = 0$$

for some coefficients c_i, then by applying σ_j to both sides, we have that

$$c_j = \sum_i c_i\delta_{ij} = \sum_i c_i\sigma_j\varphi_i = \sigma_j(\sum_i c_i\varphi_i) = \sigma_j 0 = 0.$$

Thus $c_j = 0$ for all j, so that they are linearly independent and thus a basis for P. $\square$

Thus, unisolvence not only guarantees existence of shape functions, but also uniqueness and the fact that they form a basis.

3 Finite elements on the reference triangle

Let K be a non-degenerate triangle, let P be a space of polynomials on K, and let Σ be a set of degrees of freedom (dofs) on P, which is a set of linear functionals $\sigma : P \to \mathbb{R}$.

Now let $\widehat{K}$ be the reference triangle. Then we can map $\widehat{K}$ to K via an affine linear map $T_K : \widehat{K} \to K$. Furthermore, for any polynomial $p \in P$, $p \circ T_K$ is a polynomial on $\widehat{K}$ of the same degree.

Let

$$\widehat{P} = \{p \circ T_K : p \in P\}$$

be the collection of all such polynomials on $\widehat{K}$. Since K is a non-degenerate triangle, T_K is invertible, which implies that for any $\widehat{p} \in \widehat{P}$, there is a unique $p \in P$ such that

$$\widehat{p} = p \circ T_K.$$

In other words, the map $\psi_K : P \to \widehat{P}$ defined by $\psi_K p = p \circ T_K$ is a linear isomorphism. In fact, its inverse is just given by

$$\psi_K^{-1}\widehat{p} = \widehat{p} \circ T_K^{-1}.$$

Since ψ_K is a linear isomorphism, for any $\sigma \in \Sigma$, the composition $\sigma \circ \psi_K^{-1}$ is a dof on $\widehat{P}$. Let

$$\widehat{\Sigma} = \{\sigma \circ \psi_K^{-1} : \sigma \in \Sigma\}$$

be the collection of all such dofs on $\widehat{P}$. Similar to P and $\widehat{P}$, we have that, for every $\widehat{\sigma} \in \widehat{\Sigma}$, there is a unique $\sigma \in \Sigma$ such that

$$\widehat{\sigma} = \sigma \circ \psi_K^{-1},$$

namely,

$$\sigma = \widehat{\sigma} \circ \psi_K.$$

In other words, the association

$$\sigma \in \Sigma \mapsto \sigma \circ \psi_K^{-1} \in \widehat{\Sigma}$$

is a bijection between Σ and $\widehat{\Sigma}$ (in fact, it is also a linear isomorphism, but we don't need this).

This now gives us two different triples, the triple (K, P, Σ) on K and the triple $(\widehat{K}, \widehat{P}, \widehat{\Sigma})$ on $\widehat{K}$. With how we defined everything, we have the following relationship between them.

Lemma 4. *Σ is a unisolvent set of dofs on P iff $\widehat{\Sigma}$ is a unisolvent set of dofs on $\widehat{P}$.*

Proof. Suppose that Σ is unisolvent on P. From the previous section, this means that if $p \in P$ satisfies $\sigma p = 0$ for all $\sigma \in \Sigma$, then $p = 0$. Then if $\widehat{p} \in \widehat{P}$ is such that $\widehat{\sigma}\widehat{p} = 0$ for all $\widehat{\sigma} \in \widehat{\Sigma}$, we have that

$$(\widehat{\sigma} \circ \psi_K)(\psi_K^{-1}\widehat{p}) = 0$$

for all $\widehat{\sigma} \in \widehat{\Sigma}$. From our discussion above, this means that the polynomial $p = \psi_K^{-1}\widehat{p} \in P$ satisfies

$$\sigma p = 0$$

for all $\sigma \in \Sigma$. Since Σ is unisolvent, this implies that $p = 0$, which in turn implies that $\widehat{p} = \psi_K p = 0$. Thus $\widehat{\Sigma}$ is also unisolvent. This proves one direction. The other direction follows a similar argument. $\square$

Setting things up in this way gives us more than just a relation between unisolvence. It also gives us a relation between the local shape functions on the elements.

Suppose now that (K, P, Σ) is a finite element triple as in the previous section. This means that $\dim P = n$ and $\Sigma = \{\sigma_1, \ldots, \sigma_n\}$ is a unisolvent set of dofs. From how we set everything up, this also implies that (and is in fact equivalent to) $(\widehat{K}, \widehat{P}, \widehat{\Sigma})$ is a finite element triple, with $\dim \widehat{P} = n$ and $\widehat{\Sigma} = \{\widehat{\sigma}_1, \ldots, \widehat{\sigma}_n\}$ also being a unisolvent set of dofs.

Also from the previous section, we have a unique set of local shape functions $\varphi_1, \ldots, \varphi_n$ for the finite element on K as well as a unique set of local shape functions $\widehat{\varphi}_1, \ldots, \widehat{\varphi}_n$ on $\widehat{K}$. They are related in very much the same way that the polynomials in P and $\widehat{P}$ are related.

Lemma 5. *In the setting described above, the local shape functions are related via*

$$\varphi_i \circ T_K = \widehat{\varphi}_i$$

for each i. Equivalently,

$$\psi_K \varphi_i = \widehat{\varphi}_i.$$

Proof. By uniqueness of the shape functions, it suffices to show that

$$\widehat{\sigma}_i(\varphi_j \circ T_K) = \delta_{ij}$$

for all i, j. However, by unpacking our definitions, we have that $\sigma_i = \widehat{\sigma}_i \circ \psi_K$. Therefore,

$$\delta_{ij} = \sigma_i \varphi_j = \widehat{\sigma}_i(\psi_K \varphi_j) = \widehat{\sigma}_i(\varphi_j \circ T_K),$$

which is what we wanted to show. $\square$