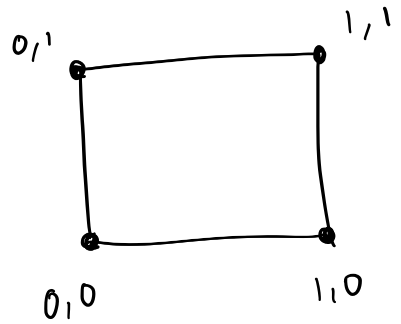
$$-\nabla\cdot\big(K(x)\nabla u(x)\big) + q(x)u(x) = f(x) \qquad x \in \mathring{\Omega} \subset \mathbb{R}^2$$

interior

$$u(x) = g(x) \text{ on } \partial\Omega$$

boundary

<u>example</u> $\Omega = [0,1]^2$ unit square

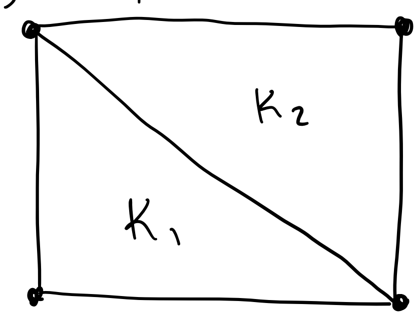


1. Mesh

T_h triangulation of Ω of size $h > 0$

<u>example</u>

$v_3 = v_{K_1,3} = v_{K_2,2}$ $\qquad$ $v_4 = v_{K_2,3}$



$v_1 = v_{K_1,1}$ $\qquad$ $v_2 = v_{K_1,2} = v_{K_2,1}$

$T_h = \{K_1, K_2\}$

local enumeration to K_1

global enumeration

local enumeration to K_2

K_i - mesh cells (triangles)

v_i - mesh vertices <u>globally</u> <u>enumerated</u>

Each mesh cell K is a triangle with 3 vertices

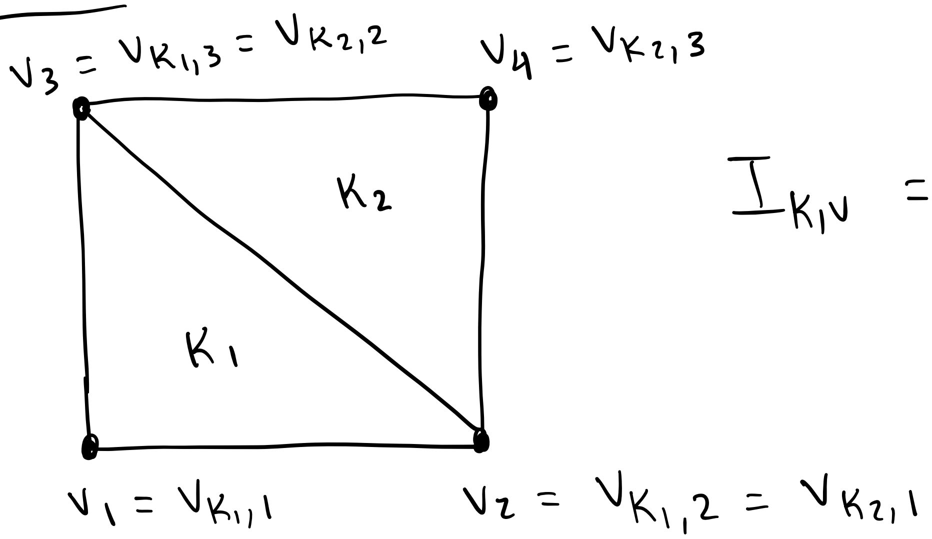
$v_{K,1}, v_{K,2}, v_{K,3}$ <u>locally</u> <u>enumerated</u>

We encode this relationship between global enumerations and local enumerations by a table

$$I_{K,V} := \begin{array}{cc} & \begin{array}{ccc} v_{K,1} & v_{K,2} & v_{K,3} \end{array} \\ \begin{array}{c} K_1 \\ K_2 \\ \vdots \\ K_N \end{array} & \underbrace{\left[\begin{array}{c} \text{global} \\ \text{vertex} \\ \text{indices} \end{array} \right]}_{\#\ \text{vertices/cell} \ = \ 3} \end{array} \left.\vphantom{\begin{array}{c} K_1 \\ K_2 \\ \vdots \\ K_N \end{array}}\right\} \#\ \text{cells}$$

cell-to-vertex enumeration

<u>Example</u>



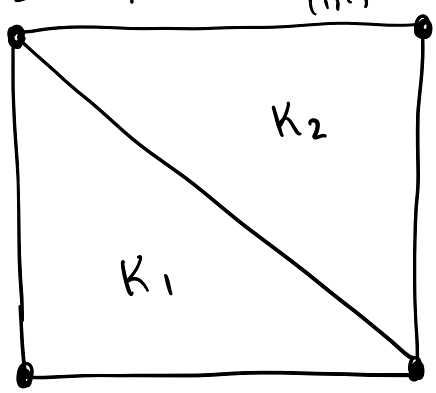
$$I_{K,V} = \begin{array}{cc} & \begin{array}{ccc} v_{K,1} & v_{K,2} & v_{K,3} \end{array} \\ \begin{array}{c} K_1 \\ K_2 \end{array} & \left[\begin{array}{ccc} 1 & 2 & 3 \\ 2 & 3 & 4 \end{array} \right] \end{array}$$

We also create a list of vertex coordinates in order of their global enumeration

$$V := \begin{array}{cc} & \begin{matrix} x & \quad & y \end{matrix} \\ \begin{matrix} v_1 \\ \vdots \\ \vdots \\ v_M \end{matrix} & \left.\underbrace{\begin{bmatrix} (x,y) \\ \text{coordinates} \\ \text{of vertices} \\ \end{bmatrix}}_{\#\text{ coordinates } = 2}\right\} \#\text{ vertices} \end{array}$$

<u>Example</u> $\Omega = [0,1]^2$

$(0,1) = v_3 = v_{K_1,3} = v_{K_2,2}$ $\qquad$ $(1,1) = v_4 = v_{K_2,3}$



$(0,0) = v_1 = v_{K_1,1}$ $\qquad$ $(1,0) = v_2 = v_{K_1,2} = v_{K_2,1}$

$$I_{K,V} = \begin{array}{cc} & \begin{matrix} v_{K,1} & v_{K,2} & v_{K,3} \end{matrix} \\ \begin{matrix} K_1 \\ K_2 \end{matrix} & \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix} \end{array}$$

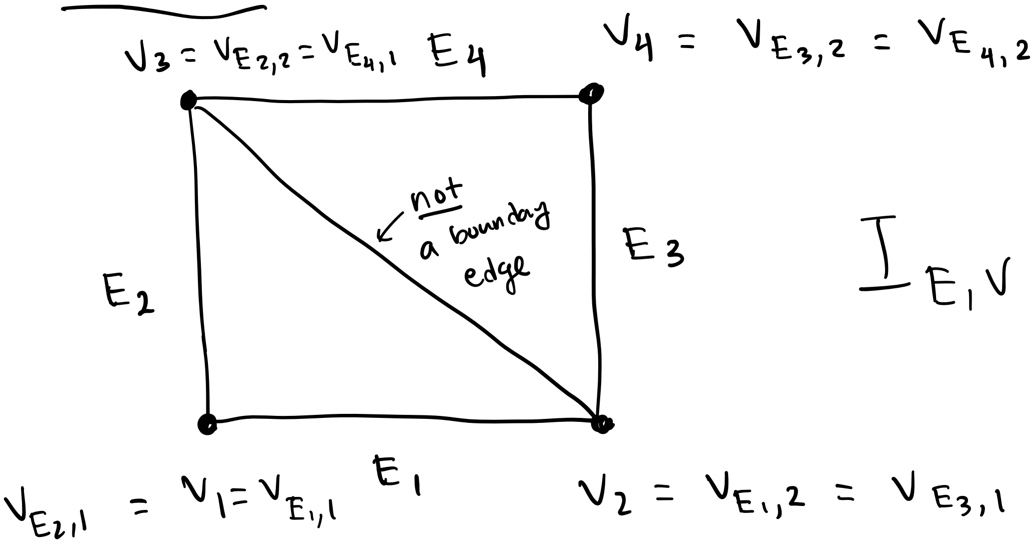
$$V = \begin{array}{cc} & \begin{matrix} x & y \end{matrix} \\ \begin{matrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{matrix} & \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \end{array}$$

$I_{K,V}$ and V are the only data structures we need when computing integrals over Ω.

If we also need the edges on the boundary of Ω, as is the case when computing integrals over $\partial\Omega$, we need an edge-to-vertex enumeration

$$I_{E,V} := \begin{array}{c} \\ E_1 \\ \vdots \\ \\ E_N \end{array} \underbrace{\begin{array}{cc} V_{E,1} & V_{E,2} \\ \left[\begin{array}{c} \text{global} \\ \text{vertex} \\ \text{enumeration} \\ \\ \end{array} \right. & \left. \right] \end{array}}_{\#\text{ vertices / edge } = 2} \left.\vphantom{\begin{array}{c} a\\b\\c\\d\\e \end{array}}\right\} \#\text{ boundary edges}$$

Example



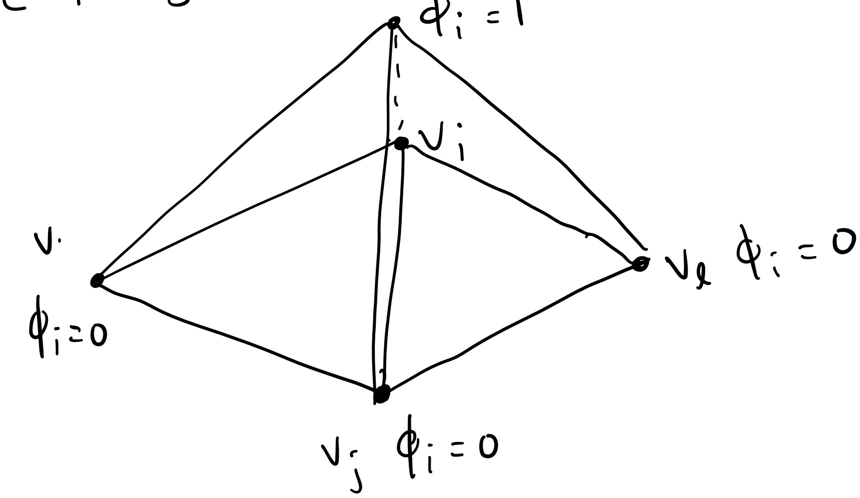
$$I_{E,V} = \begin{array}{c} \\ E_1 \\ E_2 \\ E_3 \\ E_4 \end{array} \begin{array}{c} \begin{array}{cc} V_{E,1} & V_{E,2} \end{array} \\ \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 2 & 4 \\ 3 & 4 \end{bmatrix} \end{array}$$

V, $I_{K,V}$, $I_{E,V}$ are the main data structures needed for our finite element assembly

2. Discretization

N_v - # vertices

$\{\phi_i\}_{i=1}^{N_v}$ - nodal basis functions



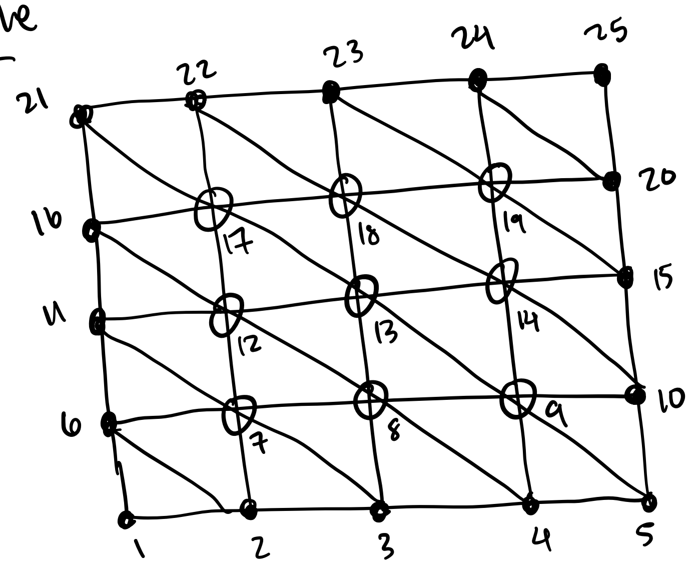
$I_v := \{1, .., N_v\}$ vertex indices

$= I_v^{\circ} \cup I_v^{\partial}$

I_v° ← interior vertices

I_v^{∂} ← boundary vertices

Example



● - boundary vertices

o - interior vertices

$I_v^{\partial} = \{1, 2, 3, 4, 5, 6, 10, 11, 15, 16, 20, 21, 22, 23, 24, 25\}$

$I_v^{\circ} = \{7, 8, 9, 12, 13, 14, 17, 18, 19\}$

Assume $\quad u = \sum_{j \in \mathcal{I}_V} u_j \phi_j \qquad u_j \in \mathbb{R}$

unknown coefficients

For <u>interior vertices</u> $i \in \mathcal{I}_V^\circ$, we

use the PDE

$$-\nabla \cdot \left(K(x) \nabla u(x) \right) + q(x) u(x) = f(x), \quad x \in \Omega^\circ \quad (*)$$

to get an equation.

Multiply $(*)$ by test function ϕ_i, $i \in \mathcal{I}_V^\circ$

and integrate-by-parts to get

$$\boxed{(1) \quad \sum_{j \in \mathcal{I}_V} \underbrace{\left(\int_\Omega K(x) \nabla \phi_j(x) \cdot \nabla \phi_i(x) + q(x) \phi_j(x) \phi_i(x) \right)}_{A_{ij}} u_j = \underbrace{\int_\Omega f(x) \phi_i(x)\, dx}_{F_i} \qquad \forall\, i \in \mathcal{I}_V^\circ}$$

The boundary term vanishes b/c when ϕ_i is at an interior vertex $i \in \mathcal{I}_V^\circ$, $\phi_i = 0$ on $\partial\Omega$.

For boundary vertices $i \in \mathcal{I}_V^\partial$, we use the boundary condition

$$u(x) = g(x) \quad x \in \partial\Omega \qquad (**)$$

to get an equation to solve.

Recalling that, since $\phi_i(v_j) = \delta_{ij} \ \forall \ i,j \in \mathcal{I}_V$, we have that $u_j = u(v_j) \ \forall \ j$. Thus, we obtain the equations

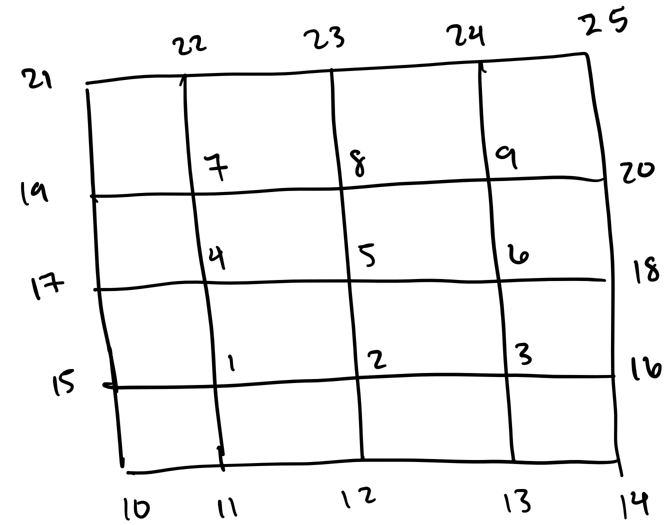
$$\boxed{(2) \qquad u_i = g(v_i) \quad \forall \ i \in \mathcal{I}_V^\partial}$$

So, we assemble the linear system corresponding to (1) and (2) and solve it to get the coefficients u_j.

If we reorder the coefficients so that

$$I_V = \{ \underbrace{1, 2, \dots, N_V^{\circ}}_{= I_V^{\circ} \text{ interior}}, \underbrace{N_V^{\circ}+1, \dots, N_V}_{= I_V^{\partial} \text{ boundary}} \}$$

For example,



Then the matrix-vector form of (1), (2) has the following block structure

$$\left[\begin{array}{c|c} \multicolumn{2}{c}{A_{ij}} \\ \hline 0 & \begin{matrix} 1 & & \\ & 1 & \\ & & \ddots \\ & & & 1 \end{matrix} \end{array}\right] \begin{array}{c} \left[\begin{array}{c} u_1 \\ \vdots \\ u_{N_V^{\circ}} \\ \hline u_{N_V^{\circ}+1} \\ \vdots \\ u_{N_V} \end{array}\right] \end{array} \begin{array}{c} I_V^{\circ} \\ \\ \\ I_V^{\partial} \end{array} = \left[\begin{array}{c} F_i \\ \\ \hline g(v_{N_V^{\circ}+1}) \\ \vdots \\ g(v_{N_V}) \end{array}\right]$$

We do <u>not</u> need to reorder the coefficients in this way. It is only done here for teaching purposes. Motivated by the block-structure above, we have the following practical algorithm for assembling the matrix-vector system from (1), (2):

1. Assemble the $N_v \times N_v$ matrix-vector system

$$\tilde{A} U = \tilde{F} \quad (*)$$

w/ $\tilde{A}_{ij} = A_{ij} \quad \forall \, i, j \in \mathcal{I}_v$.

$\tilde{F}_i = F_i$

Notice this is for <u>all the vertices</u>, not just the interior. For $i \in \mathcal{I}_v^{\circ}$, this system has the correct equations, but not for $i \in \mathcal{I}_v^{\partial}$.

2. To fix the system $(*)$, for $i \in \mathcal{I}_v^{\partial}$, we replace row i in $\tilde{A}$ by

$$\begin{array}{c} 0 \cdots 0 \; 1 \; 0 \cdots 0 \\ \uparrow \\ \text{column } i \end{array}$$

and we replace entry i in $\tilde{F}$ by $g(v_i)$.

$\boxed{\text{3. Assembly}}$

Now we need to assemble the matrix

$$A_{ij} = \int_\Omega K(x)\, \nabla\phi_j(x) \cdot \nabla\phi_i(x) + q(x)\, \phi_j(x)\phi_i(x)\, dx$$

and vector

$$F_i = \int_\Omega f(x)\, \phi_i(x)\, dx.$$

We do this by looping over the cells.

dofs/cell = 3

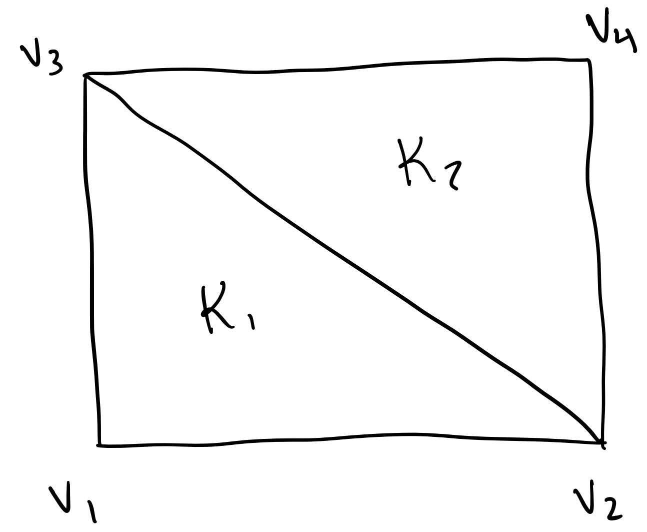
$$\overset{N_v \times N_v}{A} = \sum_K \overset{N_v \times 3}{P_K^T} \; \overset{3\times 3}{A_K} \; \overset{3 \times N_v}{P_K}$$

Note: P_K^T puts the local dofs back into the global system.

$$\overset{N_v \times 1}{F} = \sum_K \overset{N_v \times 3}{P_K^T} \; \overset{3\times 1}{F_K}$$

where P_K extracts the global dofs belonging to cell K, and A_K, F_K are the local cell system.

Example



$$u = u_1 \phi_1 + u_2 \phi_2 + u_3 \phi_3 + u_4 \phi_4$$

$$U = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \qquad P_{K_1} U = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} u_{K_1,1} \\ u_{K_1,2} \\ u_{K_1,3} \end{bmatrix}$$

$$P_{K_2} U = \begin{bmatrix} u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} u_{K_2,1} \\ u_{K_2,2} \\ u_{K_2,3} \end{bmatrix}$$

In general, $P_K U = \begin{bmatrix} u_{K,1} \\ u_{K,2} \\ u_{K,3} \end{bmatrix}$ dofs for the local vertices on K

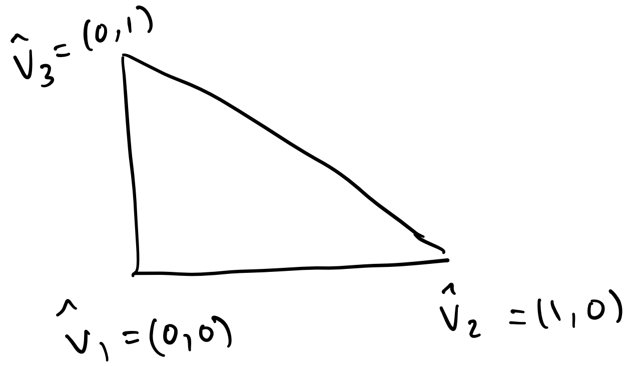
Let $\{\phi_{K,i}\}_{i=1}^{3}$ be the local basis functions on cell K. Then for $i,j = 1,2,3$

$$(A_K)_{ij} = \int_K \kappa(x) \nabla \phi_{K,j}(x) \cdot \nabla \phi_{K,i}(x) + q(x) \phi_{K,j}(x) \phi_{K,i}(x)$$

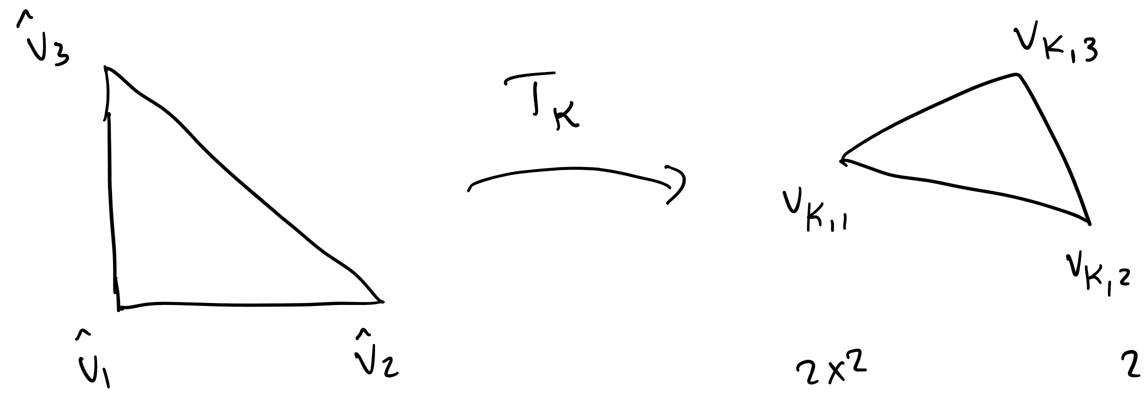
$$(F_K)_i = \int_K f(x)\,\phi_{K,i}(x)\,dx$$

We let
$$\begin{cases} \hat{\phi}_1(\hat{x},\hat{y}) := 1 - \hat{x} - \hat{y} \\ \hat{\phi}_2(\hat{x},\hat{y}) := \hat{x} \\ \hat{\phi}_3(\hat{x},\hat{y}) := \hat{y} \end{cases}$$

be the reference basis functions on the reference triangle $\hat{K} :=$



We let $T_K : \hat{K} \longrightarrow K$ be the affine linear mapping from $\hat{K}$ to K such that $T_K(\hat{v}_i) = v_{K,i}$ $\forall i$



Then
$$T_K(\hat{x},\hat{y}) = \overset{2\times 1}{v_{K,1}} + \underbrace{\overset{\substack{2\times 2\\ \text{columns}}}{\begin{bmatrix} v_{K,2}-v_{K,1} & v_{K,3}-v_{K,1} \end{bmatrix}}}_{:=\, B_K} \overset{2\times 1}{\begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix}}$$

so $DT_K := B_K$ constant 2x2 matrix

We also have that $\phi_{K,i} \circ T_K = \hat{\phi}_i \quad \forall\ i=1,2,3$

$$\nabla(\phi_{K,i} \circ T_K) = B_K^T (\nabla \phi_{K,i}) \circ T_K = \nabla \hat{\phi}_i .$$

Therefore, to compute A_K, F_K, we change coordinates back to $\hat{K}$ and obtain

$$(A_K)_{ij} = \int_{\hat{K}} \Big\{ \kappa(T_K(\hat{x})) \left(B_K^{-T} \nabla \hat{\phi}_j\right) \cdot \left(B_K^{-T} \nabla \hat{\phi}_i\right) + q(T_K(\hat{x}))\, \hat{\phi}_j(\hat{x})\, \hat{\phi}_i(\hat{x}) \Big\} |\det B_K|\, d\hat{x}$$

$$(F_K)_i = \int_{\hat{K}} f(T_K(\hat{x}))\, \hat{\phi}_i(\hat{x}) |\det B_K|\, d\hat{x} \qquad i,j = 1,2,3$$

Now, if we choose quadrature weights $\hat{\omega}_q$ and points $\hat{x}_q$ on $\hat{K}$ such that we obtain a quadrature rule

$$\int_{\hat{K}} \psi(\hat{x})\, d\hat{x} \sim \underbrace{|\hat{K}|}_{\text{area } \hat{K} = 1/2} \sum_q \psi(\hat{x}_q)\, \hat{\omega}_q$$

that is sufficiently accurate, then we compute A_K, F_K via quadrature on the reference element

$$\boxed{\begin{aligned} (A_K)_{ij} &= |\hat{K}| \sum_q \Big\{ \kappa(T_K(\hat{x}_q)) \, (B_K^{-T} \nabla \hat{\phi}_j) \cdot (B_K^{-T} \nabla \hat{\phi}_i) + \\ &\qquad q(T_K(\hat{x}_q)) \, \hat{\phi}_j(\hat{x}_q) \, \hat{\phi}_i(\hat{x}_q) \Big\} \, |\det B_K| \\ (*) \\ (F_K)_{ij} &= |\hat{K}| \sum_q f(T_K(\hat{x}_q)) \, \hat{\phi}_i(\hat{x}_q) \, |\det B_K| \end{aligned}}$$

To summarize, the following algorithm assembles the system:

For each cell K:

get local vertices $V_{K,1}, V_{K,2}\ V_{K,3}$

P_K compute $B_K := [\,V_{K,2} - V_{K,1} \quad V_{K,3} - V_{K,1}\,]$

get local dof indices $i_{K,1}, i_{K,2}, i_{K,3}$ on cell K

For each quadrature point $\hat{x}_q$ on $\hat{K}$:

compute $x_q = T_K(\hat{x}_q) = V_{K,1} + B_K \hat{x}_q$

compute $f_q = f(x_q)$

compute $K_q = K(x_q)$

For each $i = 1, 2, 3$

get global row $= i_{K,i}$

$F_K \longrightarrow$ compute $F_{i,q} = |\hat{K}|\, \hat{w}_q\, |\det B_K|\, f_q\, \hat{\phi}_i(\hat{x}_q)$

$P_K^T F_K \longrightarrow$ $F[row] += F_{iq}$

For each $j = 1, 2, 3$

get global col $= i_{K,j}$

$A_K \longrightarrow$ compute A_{ijq} like how we computed F_{iq} using ⊛ above

$P_K^T A_K \longrightarrow$ $A[row, col] += A_{ijq}$