The 1d Riemann problem

Jordan Hoffart

September 7, 2023

Contents

1 1d Riemann problem

Last time, we looked at scalar conservation laws of the form

$$\partial_t u + \nabla \cdot f(u) = 0 \text{ for all } (x,t) \in \Omega \times (0,\infty)$$
$$u(x,0) = u_0(x) \text{ for all } x \in \Omega$$

where $\Omega \subset \mathbb{R}^d$ and u_0 was some generic initial data. This time, we are going to focus on the special case when $d = 1$, $\Omega = \mathbb{R}$, and

$$u_0(x) = \begin{cases} u_L & x \leq 0 \\ u_R & x > 0 \end{cases}$$

for two states $u_L \in \mathbb{R}$ (the left state) and $u_R \in \mathbb{R}$ (the right state). Such a problem is called the *1d Riemann problem.* Our goal is to find an explicit representation formula for the solution u of the 1d Riemann problem for a fairly broad class of fluxes f.

We proceed heuristically. The first thing to observe is that if u satisfies the 1d Riemann problem

$$\partial_t u + \partial_x f(u) = 0 \text{ for all } (x,t) \in \mathbb{R} \times (0,\infty), \tag{1a}$$

$$u(x,0) = \begin{cases} u_L & x \leq 0 \\ u_R & x > 0 \end{cases} \text{ for all } x \in \mathbb{R} \tag{1b}$$

then so does $u_\lambda(x,t) := u(\lambda x, \lambda t)$ for any $\lambda > 0$. Such solutions are called *self-similar*. This suggests that we should look for solutions of the form $u(x,t) = w(x/t)$. Inserting this into our PDE gives us

$$-\left(\frac{x}{t^2}\right) w'\left(\frac{x}{t}\right) + \frac{1}{t} f'\left(w\left(\frac{x}{t}\right)\right) w'\left(\frac{x}{t}\right) = 0.$$

This in turn implies that

$$f'\left(w\left(\frac{x}{t}\right)\right) = \frac{x}{t}$$

for all $(x,t) \in \mathbb{R} \times (0,\infty)$. In particular, we have that w is a right inverse of f':

$$f'(w(\xi)) = \xi$$

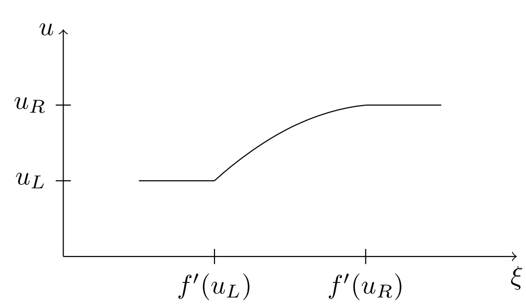
for all $\xi \in \mathbb{R}$. Conversely, if w is a right inverse for f', then setting $u(x,t) = w(x/t)$ for all $(x,t) \in \mathbb{R} \times (0,\infty)$ gives us a solution to the 1d Riemann problem (1). Therefore, solving the 1d Riemann problem is heuristically equivalent to finding a right inverse for f'. We now discuss some simple situations where f' is invertible.

2 Convex fluxes and concave fluxes

If f' is strictly monotone then it is invertible. One such case when this occurs is when f is C^2 and strictly convex. So let us assume that f has these properties. Let us also assume that $u_L < u_R$, so that $f'(u_L) < f'(u_R)$. Let us set

$$u(x,t) = \begin{cases} u_L & \frac{x}{t} \leq f'(u_L) \\ (f')^{-1}\left(\frac{x}{t}\right) & f'(u_L) \leq \frac{x}{t} \leq f'(u_R) \\ u_R & f'(u_R) \leq \frac{x}{t} \end{cases}. \tag{2}$$

A graph of u versus $\xi = x/t$ is given below.



Proposition. *For a C^2 strictly convex flux f and $u_L < u_R$, the function u defined in* (2) *is the entropy solution to the 1d Riemann problem* (1).

Proof. See [2]. ☐

We call this solution the *expansion wave* solution.

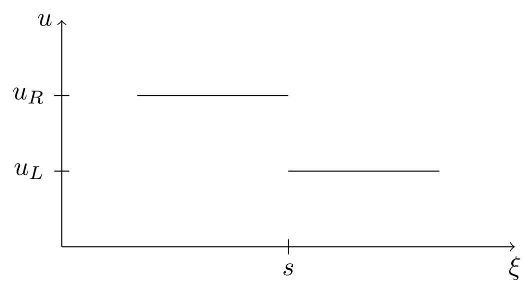
Now what if instead $u_R < u_L$? In this case, one can show [2] that the entropy solution to (1) is a traveling discontinuity given by

$$u(x,t) = \begin{cases} u_L & \frac{x}{t} \leq s \\ u_R & s < \frac{x}{t} \end{cases} \tag{3}$$

where

$$s = \frac{f(u_L) - f(u_R)}{u_L - u_R} \in \mathbb{R}. \tag{4}$$

A graph of u versus $\xi = x/t$ for (3) is given below.



The literature refers to equation (4) as the *Rankine-Hugoniot condition*. We call (3) the *shock wave* solution with *shock velocity* s. The expansion wave and shock wave solutions completely characterize the entropy solution to (1) in the case of a C^2 strictly convex flux f.

What happens instead if f is strictly concave? If we change variables $x \mapsto -x$ in (1), then we obtain a related 1d Riemann problem

$$\partial_t v + \partial_x g(v) = 0 \text{ for all } x \in \mathbb{R} \times (0, \infty)$$
$$v(x,0) = v_0(x) = \begin{cases} v_L & x < 0 \\ v_R & x \geq 0 \end{cases} \text{ for all } x \in \mathbb{R}$$

with $g = -f$, $v_L = u_R$, and $v_R = u_L$. Since g is strictly convex, we apply our results above to conclude that

1. If $u_L < u_R$, then $v_R < v_L$, so

   $$u(x,t) = v(-x,t) = \begin{cases} v_L & -\frac{x}{t} \leq \tilde{s} \\ v_R & -\frac{x}{t} > \tilde{s} \end{cases} = \begin{cases} u_L & \frac{x}{t} < -\tilde{s} \\ u_R & -\tilde{s} \leq \frac{x}{t} \end{cases}$$

   with

   $$-\tilde{s} = -\frac{g(v_L) - g(v_R)}{v_L - v_R} = \frac{f(u_R) - f(u_L)}{u_R - u_L} = s$$

   is the entropy solution.

2. If $u_R < u_L$, then $v_L < v_R$, so

$$u(x,t) = v(-x,t) = \begin{cases} v_L & -\frac{x}{t} \le g'(v_L) \\ (g')^{-1}\left(-\frac{x}{t}\right) & g'(v_L) \le -\frac{x}{t} \le g'(v_R) \\ v_R & g'(v_R) \le -\frac{x}{t} \end{cases}$$

$$= \begin{cases} u_L & \frac{x}{t} \le f'(u_L) \\ (-f')^{-1}\left(-\frac{x}{t}\right) \equiv (f')^{-1}\left(\frac{x}{t}\right) & f'(u_L) \le \frac{x}{t} \le f'(u_R) \\ u_R & f'(u_R) \le \frac{x}{t} \end{cases}$$

In other words, when f is strictly concave, the entropy solution looks just like in the strictly concave case but with the cases involving u_L and u_R reversed.

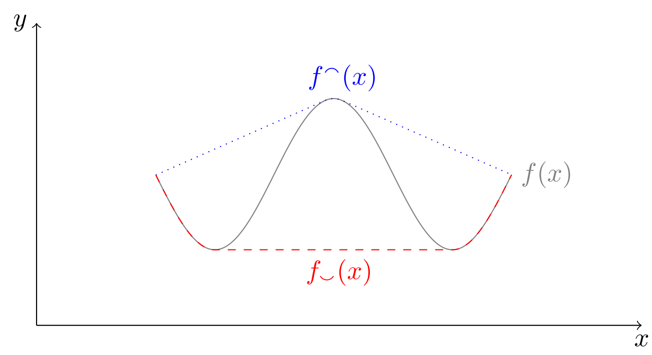
We can consolidate our cases if we introduce the following notation. The *lower convex envelope* of a function f is the function $f_\smile$ defined by

$$f_\smile(x) = \sup\{g(x) : g \text{ convex and } g \le f\}.$$

Similarly, the *upper concave envelope* of f is the function $f^\frown$ given by

$$f^\frown(x) = \inf\{g(x) : g \text{ concave and } f \le g\}.$$

An example of a function with its lower convex and upper concave envelope is given below.



Observe that, if f is convex on an interval $[a, b]$, then $f = f_\smile$ on that interval while

$$f^\frown(x) = f(a)\frac{x-b}{a-b} + f(b)\frac{x-a}{b-a}$$

on the interval. In the case $a = u_L$ and $b = u_R$, the slope of $f^\frown$ is the shock speed s. Similarly, if f is concave, then we switch $f_\smile$ and $f^\frown$. With this notation, we have that for a C^2 strictly monotone flux f:

1. If $u_L < u_R$, then the entropy solution u to the 1d Riemann problem (1) is

$$u(x,t) = \begin{cases} u_L & \frac{x}{t} \le f'_\smile(u_L) \\ (f'_\smile)\left(\frac{x}{t}\right) & f'_\smile(u_L) \le \frac{x}{t} \le f'_\smile(u_R) \\ u_R & f'_\smile(u_R) \le \frac{x}{t} \end{cases}. \tag{5}$$

2. If $u_R < u_L$, then the entropy solution u to the 1d Riemann problem (1) is

$$u(x,t) = \begin{cases} u_L & \frac{x}{t} \le (f^\frown)'(u_L) \\ (f^\frown)'\left(\frac{x}{t}\right) & (f^\frown)'(u_L) \le \frac{x}{t} \le (f^\frown)'(u_R) \\ u_R & (f^\frown)'(u_R) \le \frac{x}{t} \end{cases}. \tag{6}$$

3 The general case

Our introduction of lower convex and upper concave envelopes was more than just a notational convenience. It actually lets us describe the entropy solution to the 1d Riemann problem for any Lipschitz flux with finitely many inflection points.

Theorem (Riemann solution)**.** *Suppose that the interval with endpoints u_L and u_R can be divided into finitely many subintervals where on each subinterval*

1. *f has a bounded and continuous second derivative*
2. *f is strictly convex or strictly concave*

Then if $u_L < u_R$, the entropy solution is given by (5)*, and if $u_R < u_L$, then it is given by* (6)*.*

Proof. [1] gives the case when f is piecewise linear, while [2] and [3] give the full result. □

4 The Riemann cone and averages

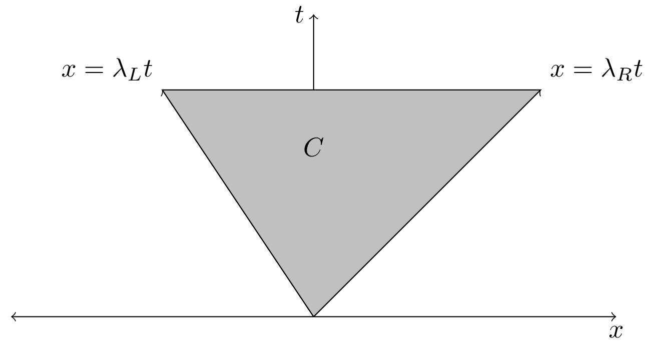
Let

$$\lambda_L = \begin{cases} f'_\smile(u_L) & u_L < u_R \\ (f^\frown)'(u_L) & u_R < u_L \end{cases} \qquad \lambda_R = \begin{cases} f'_\smile(u_R) & u_L < u_R \\ (f^\frown)'(u_R) & u_R < u_L \end{cases}$$

denote the left and right extreme wavespeeds. Then the entropy solution to the 1d Riemann problem is nontrivial inside of the *Riemann cone* defined by these wavespeeds

$$C = \left\{(x,t) \in \mathbb{R} \times (0,\infty) : \lambda_L \le \frac{x}{t} \le \lambda_R\right\}.$$

The cone is illustrated below.



Definition (Maximum wave speed)**.** Any number

$$\lambda_{\max} \geq \max\{|\lambda_L|, |\lambda_R|\}$$

is called an *upper bound on the maximum wave speed* and is denoted like so.

It's usually easier to estimate an upper bound on the maximum wave speed than to find the true values of the extreme wavespeeds themselves.

Lemma (Riemann average)**.** *Let u be the entropy solution to the 1d Riemann problem* (1)*, and let (η, q) be an entropy pair. Let the* ***Riemann average*** *be defined by*

$$\bar{u}(t) = \int_{-1/2}^{1/2} u(x,t)\, dx.$$

Let λ_{max} be an upper bound on the maximum wavespeed. Then for any $0 \leq t \leq 1/(2\lambda_{max})$,

$$\begin{aligned}\bar{u}(t) &= \frac{u_L + u_R}{2} + t(f(u_L) - f(u_R)) \\ \eta(\bar{u}(t)) &\leq \frac{\eta(u_L) + \eta(u_R)}{2} + t(q(u_L) - q(u_R))\end{aligned}$$

Proof. First, observe that for $0 \leq s \leq t$, $u(1/2, s) = u_R$ and $u(-1/2, s) = u_L$. Therefore, if we integrate (1), we get

$$\begin{aligned}0 &= \int_{-1/2}^{1/2} \int_0^t \partial_t u + \partial_x f(u)\, ds\, dx \\ &= \int_{-1/2}^{1/2} u(x,t) - u(x,0)\, dx + \int_0^t f(u(1/2,s)) - f(u(-1/2,s))\, ds \\ &= \bar{u}(t) - \frac{u_L + u_R}{2} + t(f(u_R) - f(u_L)).\end{aligned}$$

This is equivalent to what we want to show for the first equality. For the entropy inequality, we follow a similar argument to above and apply Jensen's inequality:

$$
\begin{aligned}
0 &\geq \int_0^t \int_{-1/2}^{1/2} \partial_t \eta(u) + \partial_x q(u)\, dx\, ds \\
&= \int_{-1/2}^{1/2} \eta(u(x,t)) - \eta(u(x,0))\, dx + \int_0^t q(u(1/2,s)) - q(u(-1/2,s))\, ds \\
&\geq \eta(\bar{u}(t)) - \frac{\eta(u_L) + \eta(u_R)}{2} + t(q(u_R) - q(u_L)).
\end{aligned}
$$

□

Remark (Invariant domain / maximum principle). Recall that the entropy solution u satisfies the maximum principle: $u \in \operatorname{conv}(u_L, u_R)$. The same is also true for $\bar{u}$. Therefore, the lemma above shows us that, for small enough times,

$$
\frac{u_L + u_R}{2} + t(f(u_L) - f(u_R)) \in \operatorname{conv}(u_L, u_R)
$$

as well. This important property will be used extensively later.

5 Multidimensional flux

Later, we will consider scalar conservation laws with multidimensional fluxes $f : \mathbb{R} \to \mathbb{R}^d$. In this case, we will choose a unit vector $n \in \mathbb{R}^d$ and consider the following 1d Riemann problem along the line spanned by n:

$$
\begin{aligned}
\partial_t u + \partial_x (f(u) \cdot n) = 0,& \\
u(x,0) = \begin{cases} u_L & x \leq 0, \\ u_R & x > 0 \end{cases}&.
\end{aligned}
$$

All of the theory developed here applies to this as well.

Lemma (Entropy pair). *If (η, q) is an entropy pair for the scalar conservation law with a multidimensional flux $f : \mathbb{R} \to \mathbb{R}^d$, then for each unit vector $n \in \mathbb{R}^d$, $(\eta, q \cdot n)$ is an entropy pair for the corresponding 1d Riemann problem along the line spanned by n with flux $f \cdot n$.*

Proof. Since $q_\ell' = \eta' f_\ell'$ for each $\ell \in \{1, 2, \ldots, d\}$, we have that

$$
(q \cdot n)' = (q' \cdot n) = \sum_\ell q_\ell' n_\ell = \sum_\ell \eta' f_\ell' n_\ell = \eta' (f \cdot n)'.
$$

□

6 Burgers' equation

We apply our theory to solve the 1d Riemann problem for Burgers' equation with flux $f(u) = u^2/2$. This flux is strictly convex, so if $u_L < u_R$, then the solution is given by

$$u(x,t) = \begin{cases} u_L & x \leq u_L t \\ \frac{x}{t} & u_L t \leq x \leq u_R t \\ u_R & u_R t \leq x \end{cases}$$

while if $u_R < u_L$, then the solution is given by

$$u(x,t) = \begin{cases} u_L & x \leq st \\ u_R & st \leq x \end{cases}$$

where

$$s = \frac{f(u_L) - f(u_R)}{u_L - u_R} = \frac{u_L + u_R}{2}.$$

An upper bound on the maximum wavespeed for Burgers' is then

$$\lambda_{\max} = \max\{|u_L|, |u_R|, |u_L + u_R|/2\} = \max\{|u_L|, |u_R|\}.$$

References

[1] C. Dafermos, *Polygonal approximations of solutions of the initial value problem for a conservation law*, Journal of Mathematical Analysis and Applications, (1972).

[2] H. Holden and N. H. Risebro, *Front tracking for hyperbolic conservation laws*, Springer-Verlag Berlin Heidelberg, 2007.

[3] S. Osher, *The riemann problem for nonconvex scalar conservation laws and hamilton-jacobi equations*, Proceedings of the American Mathematical Society, (1983).