Numerical PDEs

Jordan Hoffart

2023 09 08

Contents

1 Finite difference schemes for the Poisson equation in 2D **1**
- 1.1 The problem . . . 1
- 1.2 Deriving a finite difference method . . . 2
- 1.3 Approximation properties, stencils, and higher order methods . . . 3
- 1.4 Linear algebra . . . 5
- 1.5 Discrete maximum principle . . . 9
- 1.6 Well-posedness and convergence . . . 11
- 1.7 Curved boundaries . . . 13

2 The Ritz-Galerkin Method **15**
- 2.1 Sobolev Spaces . . . 16
- 2.2 Precise definitions of the strong and weak problems . . . 19
- 2.3 Galerkin's Method . . . 20
- 2.4 The Ritz method . . . 20
- 2.5 Abstract framework . . . 21

1 Finite difference schemes for the Poisson equation in 2D

1.1 The problem

Let Ω be an open and bounded domain in $\mathbb{R}^2$ with boundary $\partial\Omega$. Let $f : \Omega \to \mathbb{R}$ and $g : \Gamma = \partial\Omega \to \mathbb{R}$ be given functions. Our goal is to find a function $u : \Omega \to \mathbb{R}$ such that

$$-\Delta u = f \text{ in } \Omega, \tag{1a}$$

$$u = g \text{ on } \Gamma = \partial\Omega. \tag{1b}$$

This problem is known as the *boundary value problem for Poisson's equation.* We will not try to solve this problem analytically, but instead we will construct a related discretized problem that can be solved with some simple linear algebra.

This discrete problem will give us an approximate solution u_h to the exact solution u of the problem above.

1.2 Deriving a finite difference method

The key to discretizing (1) is to replace the *Laplacian* differential operator

$$\Delta = \partial_x^2 + \partial_y^2$$

with a suitable related difference operator Δ_h such that

$$\Delta_h \approx \Delta$$

in a suitable sense. We will also need to replace Ω and Γ by approximate discretized versions $\Omega_h \subset \Omega$ and $\Gamma_h \subset \Gamma$. The resulting discrete problem will then be to find $u_h : \Omega_h \to \mathbb{R}$ such that

$$-\Delta_h u_h = f \text{ in } \Omega_h, \tag{2a}$$

$$u_h = g \text{ on } \Gamma_h. \tag{2b}$$

This is known as the *finite difference method.*

To construct the discrete problem, we choose a mesh size $h = 1/N$, where N is a positive integer. Consider the set of *grid points* in $\mathbb{R}^2$ with spacing h

$$\mathbb{R}_h^2 = \{(mh, nh) : m, n \in \mathbb{Z}\}.$$

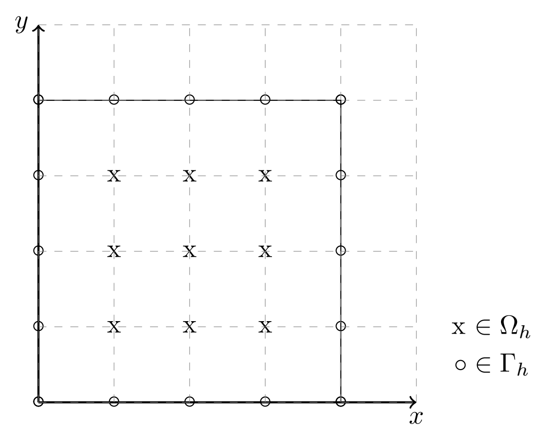
For a *node* $x = (mh, nh) \in \mathbb{R}_h^2$, its *nearest neighbors* are the nodes $((m \pm 1)h, nh)$, $(mh, (n \pm 1)h)$, and $((m \pm 1)h, (n \pm 1)h)$. We consider the set of *interior grid points*

$$\Omega_h = \mathbb{R}_h^2 \cap \Omega$$

as well as the set of *boundary grid points*

$$\Gamma_h = \{x \in \mathbb{R}_h^2 \setminus \Omega : x \text{ has a nearest neighbor in } \Omega\}.$$

For example, when $\Omega = (0,1)^2$ the unit square, the picture below shows us what Ω_h and Γ_h look like.



Remark 1. By constructing Ω_h as above, we are guaranteed that $\Omega_h \subset \Omega$. However, depending on the geometry of Γ, Γ_h need not be a subset of Γ. In what follows, we will assume that Γ is sufficiently nice such that $\Gamma_h \subset \Gamma$ (as is the case for the unit square). We will later cover curved boundaries and show how to modify our construction to handle this more general case.

To construct Δ_h, we take a smooth function u at an interior node $(x, y) \in \Omega_h$ and perform a Taylor expansion in 2D around its neighbors $(x \pm h, y)$ and $(x, y \pm h)$:

$$u(x \pm h, y) = u(x, y) \pm h\partial_x u(x, y) + \frac{h^2}{2}\partial_x^2 u(x, y) \pm \frac{h^3}{6}\partial_x^3 u(x, y) + \frac{h^4}{24}\partial_x^4 u(x, y) + O(h^5) \tag{3a}$$

$$u(x, y \pm h) = u(x, y) \pm h\partial_y u(x, y) + \frac{h^2}{2}\partial_y^2 u(x, y) \pm \frac{h^3}{6}\partial_y^3 u(x, y) + \frac{h^4}{24}\partial_y^4 u(x, y) + O(h^5) \tag{3b}$$

$$\tag{3c}$$

Adding these 4 equations together gives us

$$u(x \pm h, y) + u(x, y \pm h) = 4u(x, y) + h^2\Delta u(x, y) + \frac{h^4}{12}(\partial_x^4 + \partial_y^4)u(x, y) + O(h^5).$$

Rearranging gives us

$$\frac{u(x \pm h, y) + u(x, y \pm h) - 4u(x, y)}{h^2} = \Delta u(x, y) + \frac{h^2}{12}(\partial_x^4 + \partial_y^4)u(x, y) + O(h^3). \tag{4}$$

This says that if we call the quantity on the left $\Delta_h u(x, y)$, then for smooth functions u, Δ_h approximates Δ up to a remainder term of size h^2.

Observe that the definition

$$\Delta_h u_h(x, y) = \frac{u_h(x \pm h, y) + u_h(x, y \pm h) - 4u_h(x, y)}{h^2} \tag{5}$$

makes sense for functions $u_h : \Omega_h \to \mathbb{R}$ defined only at the grid nodes $(x, y) \in \Omega_h$. The operator Δ_h as defined in (5) is known as the *5 point approximation* to the Laplacian Δ. We will denote this particular approximation by $\Delta_h^{(5)}$.

1.3 Approximation properties, stencils, and higher order methods

We now summarize the work we did above to show that $\Delta_h^{(5)}$ is a good approximation to Δ.

Theorem 2. *If $u \in C^4(\overline{\Omega})$, then*

$$\max_{x\in\Omega_h} \|\Delta_h^{(5)} u - \Delta u\| \leq \frac{h^2}{6} \max\{\|\partial_x^4 u\|_\infty, \|\partial_y^4 u\|_\infty\} + O(h^3)$$

where

$$\|v\|_\infty := \max_{x\in\overline{\Omega}} |v(x)|.$$

Proof. Starting with a smooth function u as above and a point $(x, y) \in \Omega_h$, we do Taylor expansions around the neighboring nodes $(x \pm h, y)$ and $(x, y \pm h)$ up to the 4th derivatives. Adding them together and rearranging gives us (4) as we have shown above. Subtracting $\Delta u(x, y)$ and taking absolute values of (4), using the triangle inequality, and using the fact that

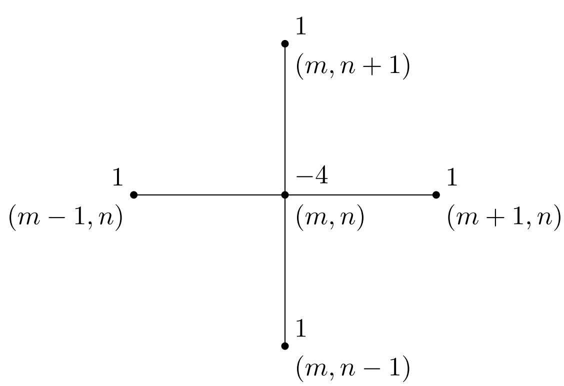
$$|a| + |b| \leq 2\max\{|a|, |b|\}$$

for any numbers a and b gives us

$$\begin{aligned} |\Delta_h^{(5)} u(x, y) - \Delta u(x, y)| &\leq \frac{h^2}{12}(|\partial_x^4 u(x, y)| + |\partial_y^4 u(x, y)|) + O(h^3) \\ &\leq \frac{h^2}{12}(\|\partial_x^4 u\|_\infty + \|\partial_y^4 u\|_\infty) + O(h^3) \\ &\leq \frac{h^2}{6}\max\{\|\partial_x^4 u\|_\infty, \|\partial_y^4 u\|_\infty\} + O(h^3). \end{aligned}$$

Since this holds for all $(x, y) \in \Omega_h$, we are done. □

Remark 3. The approximation $\Delta_h^{(5)}$ is also called the *5 point stencil*. It can be graphically represented by the following *stencil*.



Or, in a more compact matrix form:

$$\frac{1}{h^2}\begin{pmatrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{pmatrix}.$$

Remark 4. There are other higher order approximations to Δ out there. For instance, the *9 point stencil*

$$\Delta_h^{(9)} u(x,y) = \frac{1}{12h^2}(-u(x \pm 2h, y) + 16u(x \pm h, y) - u(x, y \pm 2h) + 16u(x, y \pm h) - 60u(x,y))$$

with stencil

$$\frac{1}{12h^2}\begin{pmatrix} & & -1 & & \\ & & 16 & & \\ -1 & 16 & -60 & 16 & -1 \\ & & 16 & & \\ & & -1 & & \end{pmatrix},$$

or the *compact 9 point stencil*

$$\overline{\Delta}_h^{(9)} u(x,y) = \frac{1}{6h^2}(4u(x \pm h, y) + 4u(x, y \pm h) + u(x \pm h, y \pm h) - 20u(x,y))$$

whose stencil is

$$\frac{1}{6h^2}\begin{pmatrix} 1 & 4 & 1 \\ 4 & -20 & 4 \\ 1 & 4 & 1 \end{pmatrix}$$

together with a modified right hand side

$$f_h(x,y) = f(x,y) + \frac{h^2}{12}\Delta_h^{(5)} f(x,y).$$

1.4 Linear algebra

From the non-discretized problem (1), we arrive at the discrete problem (2) whose equations hold at each of the discrete nodes (x,y) with Δ_h being the 5 point stencil (5). We will now discuss how to view these discrete equations as a matrix-vector system that can be solved with basic linear algebra techniques.

Let $x_i = ih$ and $y_j = jh$ for $i, j \in \mathbb{Z}$. Let I_h^o be the set of all *interior indices* $(i,j) \in \mathbb{Z}^2$ such that $(x_i, y_j) \in \Omega_h$, and let I_h^∂ be the set of all *boundary indices* (i,j) such that $(x_i, y_j) \in \Gamma_h$. If we let $u_{ij} = u_h(x_i, y_j)$, $f_{ij} = f(x_i, y_j)$, and $g_{ij} = g(x_i, y_j)$, we can rewrite (2) as

$$\begin{aligned} u_{i\pm1,j} + u_{i,j\pm1} - 4u_{ij} &= h^2 f_{ij} \text{ for all } (i,j) \in I_h^o \\ u_{ij} &= g_{ij} \text{ for all } (i,j) \in I_h^\partial \end{aligned}$$

Now we choose some *global enumeration* of the indices

$$(i,j) \in I_h^o \cup I_h^\partial \mapsto k = k(i,j) \in \{0, 1, \ldots, |I_h^o \cup I_h^\partial| - 1\} = M_h$$

to rewrite the equations above as

$$\begin{aligned} u_{k(i\pm1,j)} + u_{k(i,j\pm1)} - 4u_{k(i,j)} &= h^2 f_{k(i,j)} \text{ for all } k(i,j) \in M_h \text{ with } (i,j) \in I_h^o \\ u_k &= g_k \text{ for all } k = k(i,j) \in M_h \text{ with } (i,j) \in I_h^\partial \end{aligned}$$

These can be written in matrix-vector form as

$$AU = F$$

with $U = (u_k)_{k\in M_h}$ be the vector of unknowns, A being the coefficient matrix coming from the left side of the equations, and F being composed of values $h^2 f_k$, g_k coming from the right side of the equations. The exact form of A, U, and F depends on the enumeration $(i,j) \mapsto k$ as well as the domain Ω. We give an explicit example below.

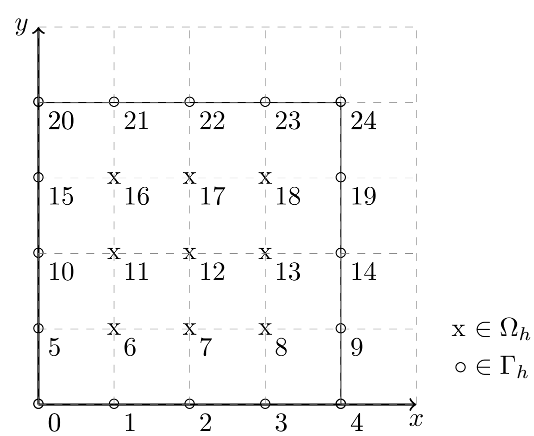
Example 5. Take $\Omega = (0,1)^2$ the unit square with $h = 1/N$. Then Ω_h consists of all nodes (x_i, y_j) with $1 \le i, j \le N-1$. Thus, $I_h^\circ = \{1, \ldots, N-1\}^2$. Similarly, Γ_h consists of all nodes of the form $(0, y_i)$, $(1, y_i)$, $(x_i, 0)$, or $(x_i, 1)$ for $0 \le i \le N$. Thus, I_h^∂ consists of all indices of the form $(0,i)$, (N,i), $(i,0)$, or (i,N) with $0 \le i \le N$. Therefore,

$$I_h^\circ \cup I_h^\partial = \{0, \ldots, N\}^2.$$

A natural global enumeration of the indices is to enumerate them by *lexicographic ordering*, which is defined by declaring indices $(i_0, j_0) < (i_1, j_1)$ if $i_0 < i_1$ or $i_0 = i_1$ and $j_0 < j_1$. This global enumeration has the following formula:

$$k(i,j) = (N+1)j + i$$

and is depicted below for $N = 4$:



Then the discrete equations with this global enumeration can be written as

$$\begin{aligned} u_{(N+1)j+i\pm 1} + u_{(N+1)(j\pm 1)+i} \\ - 4u_{(N+1)j+i} &= h^2 f_{(N+1)j+i} \text{ for all } 1 \le i, j \le N-1 \\ u_{(N+1)j+i} &= g_{(N+1)j+i} \text{ for all} \\ &\qquad (i,j) \in \{0, \ldots, N\}^2 \setminus \{1, \ldots, N-1\}^2 \end{aligned}$$

Explicitly, for $N = 4$, these equations are

$$\begin{aligned}
u_0 &= g_0 \\
u_1 &= g_1 \\
u_2 &= g_2 \\
u_3 &= g_3 \\
u_4 &= g_4 \\
u_5 &= g_5 \\
u_1 + u_5 - 4u_6 + u_7 + u_{11} &= h^2 f_6 \\
u_2 + u_6 - 4u_7 + u_8 + u_{12} &= h^2 f_7 \\
u_3 + u_7 - 4u_8 + u_9 + u_{13} &= h^2 f_8 \\
u_9 &= g_9 \\
u_{10} &= g_{10} \\
u_6 + u_{10} - 4u_{11} + u_{12} + u_{16} &= h^2 f_{11} \\
u_7 + u_{11} - 4u_{12} + u_{13} + u_{17} &= h^2 f_{12} \\
u_8 + u_{12} - 4u_{13} + u_{14} + u_{18} &= h^2 f_{13} \\
u_{14} &= g_{14} \\
u_{15} &= g_{15} \\
u_{11} + u_{15} - 4u_{16} + u_{17} + u_{21} &= h^2 f_{16} \\
u_{12} + u_{16} - 4u_{17} + u_{18} + u_{22} &= h^2 f_{17} \\
u_{13} + u_{17} - 4u_{18} + u_{19} + u_{23} &= h^2 f_{18} \\
u_{19} &= g_{19} \\
u_{20} &= g_{20} \\
u_{21} &= g_{21} \\
u_{22} &= g_{22} \\
u_{23} &= g_{23} \\
u_{24} &= g_{24}
\end{aligned}$$

In matrix-vector form:

$$A = \begin{bmatrix}
1 \\
& 1 \\
& & 1 \\
& & & 1 \\
& & & & 1 \\
& & & & & 1 \\
& 1 & & & & 1 & -4 & 1 & & & & 1 \\
& & 1 & & & & 1 & -4 & 1 & & & & 1 \\
& & & 1 & & & & 1 & -4 & 1 & & & & 1 \\
& & & & & & & & & 1 \\
& & & & & & & & & & 1 \\
& & & & & & 1 & & & & 1 & -4 & 1 & & & & 1 \\
& & & & & & & 1 & & & & 1 & -4 & 1 & & & & 1 \\
& & & & & & & & 1 & & & & 1 & -4 & 1 & & & & 1 \\
& & & & & & & & & & & & & & 1 \\
& & & & & & & & & & & & & & & 1 \\
& & & & & & & & & & & 1 & & & & 1 & -4 & 1 & & & & 1 \\
& & & & & & & & & & & & 1 & & & & 1 & -4 & 1 & & & & 1 \\
& & & & & & & & & & & & & 1 & & & & 1 & -4 & 1 & & & & 1 \\
& & & & & & & & & & & & & & & & & & & 1 \\
& & & & & & & & & & & & & & & & & & & & 1 \\
& & & & & & & & & & & & & & & & & & & & & 1 \\
& & & & & & & & & & & & & & & & & & & & & & 1 \\
& & & & & & & & & & & & & & & & & & & & & & & 1 \\
& & & & & & & & & & & & & & & & & & & & & & & & 1
\end{bmatrix}$$

$$U = (u_0 \ u_1 \ \cdots \ u_{24})^T$$

$$F = \begin{bmatrix} g_0 \\ g_1 \\ g_2 \\ g_3 \\ g_4 \\ g_5 \\ h^2 f_6 \\ h^2 f_7 \\ h^2 f_8 \\ g_9 \\ g_{10} \\ h^2 f_{11} \\ h^2 f_{12} \\ h^2 f_{13} \\ g_{14} \\ g_{15} \\ h^2 f_{16} \\ h^2 f_{17} \\ h^2 f_{18} \\ g_{19} \\ g_{20} \\ g_{21} \\ g_{22} \\ g_{23} \\ g_{24} \end{bmatrix}$$

1.5 Discrete maximum principle

The 5 point stencil $\Delta_h^{(5)}$ satisfies the following discrete maximum principle, which is a discrete analogue of a similar result for Δ.

Theorem 6. *Let $v_h : \overline{\Omega_h} \to \mathbb{R}$ satisfy $\Delta_h^{(5)} v_h \geq 0$ on Ω_h. Then*

$$\max_{x \in \Omega_h} v_h(x) \leq \max_{x \in \Gamma_h} v_h(x),$$

and equality holds iff v_h is constant.

Proof. We will first show that

$$\max_{x \in \Omega_h} v_h(x) \geq \max_{x \in \Gamma_h} v_h(x) \implies v_h \text{ is constant.}$$

Suppose that

$$\max_{x \in \Omega_h} v_h(x) \geq \max_{x \in \Gamma_h} v_h(x),$$

so that

$$\max_{x \in \overline{\Omega_h}} v_h(x) = \max_{x \in \Omega_h} v_h(x).$$

Let $x_0 \in \Omega_h$ be a node where

$$v_h(x_0) = \max_{x \in \overline{\Omega_h}} v_h(x).$$

Let x_1, x_2, x_3, and x_4 be the 4 nearest neighbors of x_0. Then

$$\begin{aligned} 0 &\leq \Delta_h^{(5)} v_h(x_0) \\ &= \frac{1}{h^2} \left(\sum_{i=1}^{4} v_h(x_i) - 4v_h(x_0) \right). \end{aligned}$$

This implies that

$$4v_h(x_0) \leq \sum_{i=1}^{4} v_h(x_i).$$

Therefore, since each $v_h(x_i) \leq v_h(x_0)$,

$$\begin{aligned} \sum_{i=1}^{4} v_h(x_i) &\leq 4v_h(x_0) \\ &\leq \sum_{i=1}^{4} v_h(x_i), \end{aligned}$$

so that actually

$$\max_{x \in \overline{\Omega_h}} v_h(x) = v_h(x_0) = \frac{1}{4} \sum_{i=1}^{4} v_h(x_i).$$

This says that $v_h(x_0)$ is the average of its values at the neighboring nodes. However, since $v_h(x_0)$ is the maximum value, this is only possible if

$$v_h(x_0) = v_h(x_1) = v_h(x_2) = v_h(x_3) = v_h(x_4),$$

so that all 5 nodes x_0, x_1, x_2, x_3, and x_4 give the maximum. We can iterate this argument through the finitely many nodes in our mesh to conclude that v_h is constant on the mesh nodes.

Thus,

$$\max_{x \in \Omega_h} v_h(x) > \max_{x \in \Gamma_h} v_h(x) \implies v_h \text{ is constant},$$

which is a contradiction, so it must be the case that

$$\max_{x \in \Omega_h} v_h(x) \leq \max_{x \in \Gamma_h} v_h(x).$$

Also, from our observation above,

$$\max_{x \in \Omega_h} v_h(x) = \max_{x \in \Gamma_h} v_h(x) \iff v_h \text{ is constant}.$$

$\square$

Remark 7. The previous theorem also holds for other stencils Δ_h^* that are

1. *diagonally dominant*: the sum of the absolute values of the off-diagonal elements is less than the absolute value of the diagonal element

2. *definite type*: the diagonal element is negative while the off-diagonal elements are positive

1.6 Well-posedness and convergence

Theorem 8. *The discrete problem*

$$\begin{aligned} -\Delta_h^{(5)} u_h &= f \text{ in } \Omega_h \\ u_h &= g \text{ on } \Gamma_h \end{aligned}$$

has a unique solution $u_h : \overline{\Omega_h} \to \mathbb{R}$. *Furthermore,*

$$\max_{x \in \overline{\Omega_h}} |u_h(x)| \leq \frac{1}{8} \max_{x \in \Omega_h} |f(x)| + \max_{x \in \Gamma_h} |g(x)|.$$

Proof. The discrete problem is a square system of linear equations in the unknowns $u_h(x)$ where $x \in \Omega_h \cup \Gamma_h = \overline{\Omega_h}$. Therefore, it suffices to show that if u_h satisfies the homogenous problem with $f = g = 0$, then $u_h = 0$. Indeed, if u_h satisfies the homogenous problem, then we have that

$$\Delta_h^{(5)} u_h = 0$$

on $\overline{\Omega_h}$. Then the maximum principle implies that

$$\max_{x \in \overline{\Omega_h}} u_h(x) \leq 0.$$

Similarly, we also have that

$$\Delta_h^{(5)}(-u_h) = 0,$$

so we once again apply the maximum principle to conclude that

$$\max_{x \in \overline{\Omega_h}} (-u_h(x)) \leq 0.$$

Therefore, for any $x \in \overline{\Omega_h}$,

$$|u_h(x)| = \begin{cases} u_h(x) & \text{if } 0 > -u_h(x) \\ -u_h(x) & \text{if } 0 > u_h(x) \end{cases}.$$

In either case,

$$|u_h(x)| \leq \max\{\max_{x \in \overline{\Omega_h}} u_h(x), \max_{x \in \overline{\Omega_h}(-u_h(x))} \leq 0,$$

so that

$$u_h = 0.$$

This proves existence and uniqueness to the non-homogenous problem as desired.

Now for the inequality, we first observe that if p is a polynomial in 2 variables of degree 2, then

$$\Delta_h^{(5)} p = \Delta p.$$

With this in mind, we let

$$p(x, y) = \frac{(x - 1/2)^2 + (y - 1/2)^2}{4}.$$

Then

1. $0 \leq p \leq 1/8$
2. $\Delta_h^{(5)} p = \Delta p = 1.$

Let

$$w_h(x, y) = p(x, y)\|f\|_\infty$$

for all $(x, y) \in \overline{\Omega_h}$. Then from our properties on p, we have that $\Delta_h^{(5)} w_h = \|f\|_\infty$ and $0 \leq w_h \leq \frac{1}{8}\|f\|_\infty$ on $\overline{\Omega_h}$. It follows from this and our existence/uniqueness result that

$$\Delta_h^{(5)}(u_h + w_h) = f + \|f\|_\infty \geq 0$$

and

$$\Delta_h^{(5)}(-u_h + w_h) = -f + \|f\|_\infty \geq 0$$

on Ω_h. The maximum principle then says that

$$\begin{aligned}
\max_{x\in\Omega_h}(\pm u_h) &\leq \max_{x\in\Omega_h}(\pm u_h + w_h) \\
&\leq \max_{x\in\Gamma_h}(\pm u_h + w_h) \\
&\leq \max_{x\in\Gamma_h}|u_h(x)| + \max_{x\in\Gamma_h} w_h(x) \\
&\leq \|g\|_{\infty,\Gamma} + \frac{1}{8}\|f\|_\infty.
\end{aligned}$$

This implies the inequality that we want to show. □

Theorem 9. *Let $u \in C^4(\overline{\Omega})$ be the solution to the continuous problem* (1) *and let u_h be the solution to the discrete problem* (2) *with the 5 point stencil. Then*

$$\|u - u_h\|_{\infty,\overline{\Omega_h}} \leq \frac{1}{8}\|\Delta u - \Delta_h^{(5)} u\|_{\infty,\Omega_h} \leq \frac{h^2}{48}\max\{\|\partial_x^4 u\|_\infty, \|\partial_y^4 u\|_\infty\} + O(h^3).$$

Proof. Let $e_h = u - u_h$ on $\overline{\Omega_h}$. Then $e_h = 0$ on Γ_h and

$$\begin{aligned}\Delta_h^{(5)} e_h &= \Delta_h^{(5)} u - \Delta_h^{(5)} u_h \\ &= \Delta_h^{(5)} u + f \\ &= \Delta_h^{(5)} u - \Delta u\end{aligned}$$

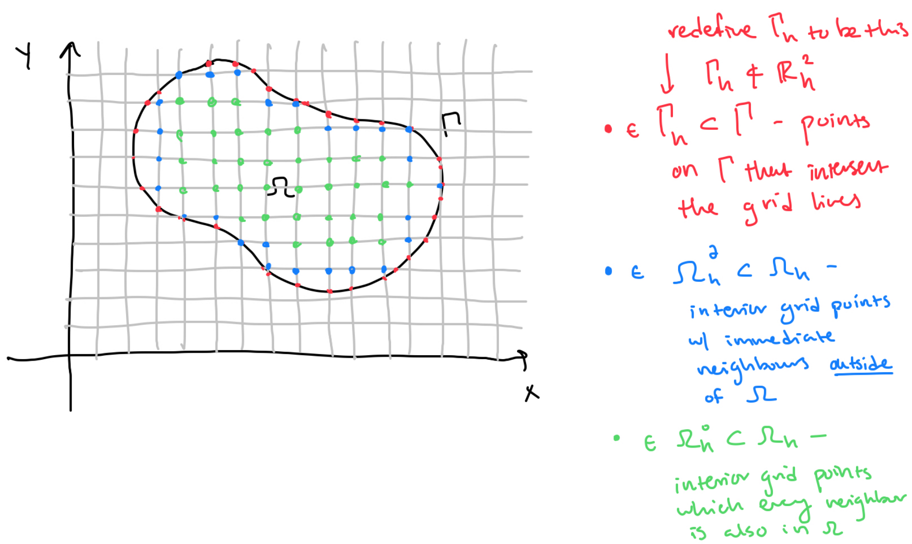
on Ω_h. Thus e_h satisfies the assumptions of the previous theorem with the right hand side f replaced by $\Delta_h^{(5)} u - \Delta u$ and g replaced by 0. Hence,

$$\|e_h\|_{\infty,\overline{\Omega_h}} \leq \frac{1}{8} \|\Delta_h^{(5)} u - \Delta u\|_{\infty,\Omega_h}.$$

This proves the first inequality. The last inequality follows from Theorem 2. □

1.7 Curved boundaries

Our analysis so far works for the assumption that we can place our mesh nodes exactly on the boundary Γ so that $\Gamma_h \subset \Gamma \cap \mathbb{R}_h^2$. What if this is not possible, as is often the case when dealing with curved boundaries? We draw a picture below as an example.



as in the picture above, we define the following sets:

1. $\Gamma_h \subset \Gamma$ as the set of points on Γ that intersect the grid axes. That is, $x \in \Gamma_h$ iff $x \in \Gamma$ and there is $y \in \Omega_h$ such that x lies on the straight line connecting y to one of its immediate neighbors. Observe in this case that Γ_h is not necessarily a subset of $\mathbb{R}_h^2$, so that points in Γ_h are not necessarily grid nodes.

2. $\Omega_h^\partial \subset \Omega_h$ the set of all interior nodes with at least one immediate neighbor lying outside of $\overline{\Omega}$.

3. $\Omega_h^\circ \subset \Omega_h$ the set of all interior nodes whose immediate neighbors lie inside of $\overline{\Omega}$.

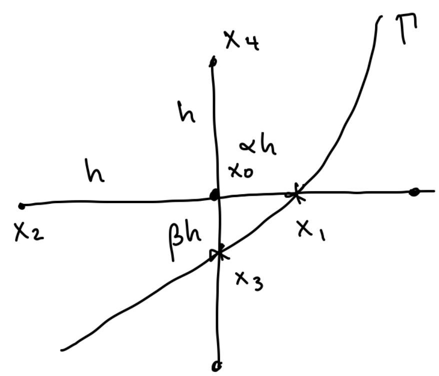
For interior nodes in Ω_h°, nothing needs to change from our usual finite difference discretization above. However, since points in Ω_h^∂ have neighboring nodes that do not lie in either Ω_h or Γ, we need to do something different. We have a few options.

1. (Constant approximation). For each $x \in \Omega_h^\partial$, we can impose the condition that $u_h(x) = g(x_*)$, where $x_* \in \Gamma_h$ lies on (one of) the lines connecting x to one of its neighbors outside of $\overline{\Omega}$.

2. (Linear approximation). For each $x \in \Omega_h^\partial$, we can require that $u_h(x)$ is the linear interpolation of $u_h(x_0)$ and $g(x_*)$, where x_* is as above and x_0 is the opposite neighbor that lies in Ω_h. Explicity, if the distance from x to x_* is αh with $0 < \alpha < 1$, then we have that

$$u_h(x) = \frac{h}{h+\alpha h} u_h(x_0) + \frac{\alpha h}{h + \alpha h} g(x_*) = \frac{1}{1+\alpha}(u_h(x_0) + \alpha g(x_*)).$$

3. (Shortley-Weller approximation). For each $x \in \Omega_h^\partial$, we modify the 5 point stencil to include the boundary points when needed. For example,



we use the modified 5 point stencil at the node x_0 in the picture above:

$$\Delta_h^{SW} u_h(x_0) = \frac{2}{h^2}\left(\frac{1}{\alpha(1+\alpha)} u_h(x_1) + \frac{1}{1+\alpha} u_h(x_2) \right.$$
$$\left. + \frac{1}{\beta(1+\beta)} u_h(x_3) + \frac{1}{1+\beta} u_h(x_3) - \left(\frac{1}{\alpha} + \frac{1}{\beta}\right) u_h(x_0)\right)$$

Theorem 10. *For the 5 point stencil with boundary approximation, we get*

$$\|u - u_h\|_{\infty,\overline{\Omega_h}} \leq \frac{1}{24} d(\Omega)^2 C h^2 + \begin{cases} O(h) & \text{constant} \\ O(h^2) & \text{linear} \\ O(h^3) & \text{Shortley-Weller} \end{cases}$$

where $d(\Omega)$ is the diameter of the domain Ω and C is a constant that does not depend on h.

2 The Ritz-Galerkin Method

We start with the (modified) Poisson equation: Find $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ such that

$$-\Delta u + \gamma(x) u = f(x) \text{ in } \Omega, \tag{6a}$$

$$u(x) = 0 \text{ on } \Gamma = \partial\Omega. \tag{6b}$$

We assume that $\gamma(x) \geq 0$ and $f(x)$ are given functions. To derive the Ritz-Galerkin method, we multiply our PDE by a smooth function $\varphi : \Omega \to \mathbb{R}$ that is 0 on the boundary and integrate over Ω:

$$-\int_\Omega \varphi \Delta u \, dx + \int_\Omega \gamma \varphi u \, dx = \int_\Omega \varphi f \, dx$$

We recall the following product rule from calculus:

$$\begin{aligned} \nabla \cdot (\varphi \nabla u) &= \sum_i \partial_i (\varphi \partial_i u) \\ &= \sum_i (\partial_i \varphi)(\partial_i u) + \varphi \partial_i^2 u \\ &= (\nabla \varphi) \cdot (\nabla u) + \varphi \Delta u. \end{aligned}$$

Rearranging gives us

$$-\varphi \Delta u = (\nabla \varphi) \cdot (\nabla u) - \nabla \cdot (\varphi \nabla u).$$

Then if we integrate over Ω and use the divergence theorem, we get the following integration-by-parts formula:

$$-\int_\Omega \varphi \Delta u \, dx = \int_\Omega (\nabla \varphi) \cdot (\nabla u) \, dx - \int_\Gamma \varphi \nabla u \cdot n \, ds.$$

Using this formula above along with the fact that $\varphi = 0$ on Γ gives us

$$\int_\Omega (\nabla \varphi) \cdot (\nabla u) + \gamma \varphi u \, dx = \int_\Omega \varphi f \, dx \tag{7}$$

What we have shown is that if we have a smooth solution u to the *strong problem* (6), then u solves the *weak problem*: find $u : \Omega \to \mathbb{R}$ such that (7) holds for all smooth functions $\varphi : \Omega \to \mathbb{R}$ that are 0 on Γ. Such functions φ are called *test functions*.

2.1 Sobolev Spaces

We call (6) the strong problem because, in order for the equations to be well-defined, we need u to be at least twice differentiable in Ω and to be zero on the boundary. However, the equation in the weak problem (7) is well-defined for a much larger, less regular class of functions that u can belong to. We define this class of functions below. The functions spaces we define are part of a broad class of spaces called *Sobolev spaces*, named after the Russian mathematician Sergei Sobolev who first studied them in the mid 1900's.

Definition 11 ($L^2(\Omega)$)**.** We let $L^2(\Omega)$ be the space of all functions $\varphi : \Omega \to \mathbb{R}$ that are *square integrable*. That is, all φ such that φ^2 is integrable and

$$\int_\Omega \varphi^2 \, dx < \infty.$$

Remark 12. For the more advanced student, the notion of integration in the previous definition is with respect to the Lebesgue measure on $\mathbb{R}^d$. For the student that has not seen this before, it is safe to think of this as the more familiar Riemann integration without being too far wrong.

Remark 13. Also, for the advanced student that has encountered L^p spaces before, we remember that $L^p(\Omega)$ is actually the collection of all equivalence classes of p-integrable functions under the equivalence of being equal almost everywhere. The novice student can forget this without any harm.

Before we list some of the main properties of $L^2(\Omega)$, we recall some basic facts and definitions from linear algebra and functional analysis:

Definition 14. An *inner product* on a vector space V is a map $(\cdot, \cdot) : V \times V \to \mathbb{R}$ that is

1. (Linear on the left) $(\alpha f + g, h) = \alpha(f, h) + (g, h)$ for all $\alpha \in \mathbb{R}$, $f, g \in V$.
2. (Symmetric) $(f, g) = (g, f)$ for all $f, g \in V$.
3. (Positive definite) $(f, f) > 0$ if $f \in V$ and $f \neq 0$.

A vector space V paired with an inner product defined on it is called an *inner product space*.

Definition 15. A *norm* on an vector space V is a map $\| \cdot \| : V \to [0, \infty)$ that

1. (is homogeneous) $\|\alpha f\| = |\alpha| \|f\|$ for all $\alpha \in \mathbb{R}$ and all $f \in V$.
2. (is positive definite) $\|f\| > 0$ if $f \in V$ and $f \neq 0$
3. (satisfies the triangle inequality) $\|f + g\| \leq \|f\| + \|g\|$ for all $f, g \in V$.

A vector space V paired with a norm on it is called a *normed vector space*.

Proposition 16. *If $(\cdot,\cdot)$ is an inner product on a vector space V, then the map*

$$\|f\| = \sqrt{(f,f)}$$

is a norm on V, called the induced norm.

Definition 17. A sequence (f_n) in a normed vector space V *converges* to $f \in V$ if the sequence $\|f_n - f\| \to 0$ in $\mathbb{R}$.

Definition 18. A sequence (f_n) in a normed vector space V is *Cauchy* if, for all $\varepsilon > 0$, there is $N > 0$ such that when $m, n > N$, $\|f_m - f_n\| < \varepsilon$. Heuristically, a sequence is Cauchy if its tails get arbitrarily close to one another.

Remark 19. In a normed space V, there may be Cauchy sequences that do not converge to anything in V.

Definition 20. A normed vector space V is *complete* if every Cauchy sequence in V converges in V. A complete normed vector space is also known as a *Banach space.*

Definition 21. An inner product space that is complete under its induced norm is known as a *Hilbert space.*

Now we state the main property of $L^2(\Omega)$ that we need.

Proposition 22. *The space $L^2(\Omega)$ is a Hilbert space under the inner product*

$$(f,g)_{L^2(\Omega)} = \int_\Omega fg\,dx.$$

We also need a couple more spaces that are based on the $L^2(\Omega)$ space. The first one is the space $H^1(\Omega)$, defined as follows.

Definition 23. The space $H^1(\Omega)$ is the space of all functions $\varphi \in L^2(\Omega)$ such that $\nabla\varphi \in L^2(\Omega)^d$ as well.

Remark 24. A function $\varphi \in L^2(\Omega)$ need not be differentiable in the usual sense, but there is an extension of the usual derivative called a *distributional derivative*, which is always defined for a function in $L^2(\Omega)$. We denote it by the same symbol $\nabla\varphi$ that we also use for the classical derivative. In general, the distributional derivative is a much more exotic object than a function defined from Ω to $\mathbb{R}^d$, but when $\nabla\varphi \in L^2(\Omega)^d$, then we say that $\varphi \in H^1(\Omega)$. In this case, we say that $\nabla\varphi$ is the *weak derivative* of φ. We will not go into too much detail with distributional derivatives, and the novice student can get by on a first reading with thinking of the weak derivative as the usual derivative (with a few minor caveats that will be covered later).

Proposition 25. *$H^1(\Omega)$ is a strict subset of $L^2(\Omega)$, and $H^1(\Omega)$ is* not *complete with respect to the L^2 norm. However, $H^1(\Omega)$* is *complete with respect to the norm induced from the inner product*

$$(f,g)_{H^1(\Omega)} = (f,g)_{L^2(\Omega)} + (\nabla f, \nabla g)_{L^2(\Omega)^d}.$$

Remark 26. The map

$$(f,g) \in H^1(\Omega)^2 \mapsto (\nabla f, \nabla g)_{L^2(\Omega)^d}$$

is *not* an inner product on $H^1(\Omega)$. It satisfies every property except being positive-definite. Such maps are called *semi-inner products*. The induced map

$$|f|_{H^1(\Omega)} := \sqrt{(\nabla f, \nabla f)_{L^2(\Omega)}}$$

is *not* a norm on $H^1(\Omega)$. Once again, it satisfies every property except being positive definite. Such maps are called *seminorms*.

The next space we need is the space $H_0^1(\Omega)$, which incorporates boundary conditions.

Definition 27. The space $H_0^1(\Omega)$ is the space of all functions $\varphi \in H^1(\Omega)$ such that $\varphi = 0$ on Γ.

Remark 28. For the advanced student that knows some measure theory, since $H^1(\Omega) \subset L^2(\Omega)$, a member φ of $H^1(\Omega)$ is really an equivalence class of functions that have been identified under almost everywhere equality. Furthermore, since Γ is the boundary of an open subset of $\mathbb{R}^d$, it is of measure 0. Thus we can redefine any function $\varphi \in H^1(\Omega)$ on Γ without changing its equivalence class. Thus, the statement $\varphi = 0$ on Γ is not well-defined.

There is a way to make this notion more rigorous by introducing the concept of *trace operators* that correspond functions defined φ on Ω with functions ψ defined on Γ such that ψ can be thought of as the function φ restricted to Γ. However, this is beyond the scope of these notes, and the novice student should just think of functions in $H^1(\Omega)$ as having well-defined values on Γ without worrying too much about what's really going on.

Definition 29. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two different norms on a vector space V. We say that the norms are *equivalent* if there are constants $c_1, c_2 > 0$ such that

$$c_1\|v\|_1 \leq \|v\|_2 \leq c_2\|v\|_1$$

for all $v \in V$.

Proposition 30. *Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two equivalent norms on a vector space V. Then a sequence (v_n) in V converges in one norm iff it converges in the other norm, and it is Cauchy in one norm iff it is Cauchy in the other norm. Therefore, V is complete with respect to one norm iff it is complete with respect to the other norm.*

Proposition 31. *The semi-inner product $(f,g) \mapsto (\nabla f, \nabla g)_{L^2(\Omega)^2}$ is actually an inner product on $H_0^1(\Omega)$. Thus, for $f, g \in H_0^1(\Omega)$, we let*

$$(f,g)_{H_0^1(\Omega)} = (\nabla f, \nabla g)_{L^2(\Omega)^d}.$$

Furthermore, the corresponding induced norm on $H_0^1(\Omega)$

$$\|f\|_{H_0^1(\Omega)} = |f|_{H^1(\Omega)} = \|\nabla f\|_{L^2(\Omega)^d}$$

is equivalent to the H^1 norm on $H_0^1(\Omega)$.

To finish this subsection, we state some approximation results that essentially says that we only need to work with smooth functions whenever we work with Sobolev spaces. First, we need a definition.

Definition 32. A subspace U of a normed vector space V is *dense* in V if, for each $v \in V$, there is a sequence $(u_n) \subset U$ that converges to $v \in V$.

Proposition 33. *The space $C^\infty(\Omega)$ is dense in $L^2(\Omega)$ with the L^2 norm. It is also dense in $H^1(\Omega)$ with the H^1 norm. The space $C_0^\infty(\Omega)$ of all smooth functions that are 0 on Γ is dense in $H_0^1(\Omega)$ with the H_0^1 norm.*

2.2 Precise definitions of the strong and weak problems

We can now give precise definitions of the strong and weak problems. In what follows, we assume $f \in L^2(\Omega)$ and $\gamma(x) \geq 0$ is continuous and bounded on Ω.

Definition 34. The *strong problem* is to find $u \in C^2(\Omega) \cap C(\overline{\Omega})$ such that

$$\begin{aligned} -\Delta u + \gamma(x)u &= f(x) \text{ on } \Omega \\ u &= 0 \text{ on } \Gamma \end{aligned}$$

Definition 35. The *weak problem* is to find $u \in H_0^1(\Omega)$ such that

$$(\nabla\varphi, \nabla u)_{H_0^1(\Omega)} + (\gamma\varphi, u)_{L^2(\Omega)} = (\varphi, f)_{L^2(\Omega)}$$

for all $\varphi \in H_0^1(\Omega)$.

We can also more precisely show how these problems are related to one another.

Proposition 36. *If $u \in C^2(\Omega) \cap C(\overline{\Omega})$ solves the strong problem, then $u \in H_0^1(\Omega)$ and u solves the weak problem.*

Proof. Since u is smooth and is 0 on the boundary, $u \in H_0^1(\Omega)$. Now we multiply the PDE in the strong problem by a smooth test function $\varphi \in C_0^\infty(U)$ and integrate by parts to get that u satisfies the equation in the weak problem for all functions $\varphi \in C_0^\infty(\Omega)$. Since $C_0^\infty(\Omega)$ is dense in $H_0^1(\Omega)$, we conclude that u also solves the weak problem. ☐

This shows that the strong problem implies the weak problem in a certain sense. We can also prove the reverse implication, but we need the following lemma.

Lemma 37 (Fundamental lemma of variational calculus)**.** *If a continuous function v satisfies*

$$\int_\Omega \varphi v \, dx = 0$$

for all $\varphi \in C_0^\infty(\Omega)$, then $v = 0$ on Ω.

Proposition 38. *If f and γ are continuous, and if $u \in H_0^1(\Omega)$ solves the weak problem and also happens to be more smooth in the sense that $u \in C^2(\Omega) \cap C(\overline{\Omega})$, then u solves the strong problem.*

Proof. Let $\varphi \in C_0^\infty(\Omega)$. Then since u is classically smooth, we can start with the equation in the weak problem, undo our integration by parts, and arrive at

$$\int_\Omega \varphi(-\Delta u + \gamma u)\, dx = \int_\Omega f \varphi\, dx.$$

Subtract the integral on the right from both sides to get

$$\int_\Omega \varphi v = 0$$

for all $\varphi \in C_0^\infty(\Omega)$, where $v = -\Delta u + \gamma u - f$ is continuous on Ω. The fundamental lemma of variational calculus finishes the proof. $\square$

2.3 Galerkin's Method

We now present an approach on how to discretize the weak problem, called *Galerkin's method.* We choose a sequence $V_{h_0} \subset V_{h_1} \subset \cdots \subset H_0^1(\Omega)$ of **finite dimensional** subspaces of $H_0^1(\Omega)$, and on each subspace V_h, we solve the *discrete weak problem*: find $u_h \in V_h$ such that

$$(\varphi_h, u_h)_{H_0^1(\Omega)} + (\gamma \varphi_h, u_h)_{L^2(\Omega)} = (\varphi_h, f)_{L^2(\Omega)}$$

for all $\varphi_h \in V_h$. The spaces V_h should be chosen in a way that such that, as $h \to 0$, $u_h \to u$ in a suitable sense.

2.4 The Ritz method

The Galerkin discretization of the weak problem by posing it on a finite dimensional subspace is intuitive, but is not so easy to work with in terms of proving that such discrete solutions u_h exist and converge to the continuous solution u. For that, we look at an alternative formulation called the *Ritz method* or the *Ritz projection.* The Ritz method is to look at the following *minimization problem*: find $u \in H_0^1(\Omega)$ that minimizes the following functional $E : H_0^1(\Omega) \to \mathbb{R}$ defined by

$$E(\varphi) = \frac{1}{2}(\|\varphi\|^2_{H_0^1(\Omega)} + (\gamma\varphi, \varphi)_{L^2(\Omega)}) - (\varphi, f)_{L^2(\Omega)}$$

We project this minimization problem onto one of the finite dimensional subspaces V_h chosen as above and look at the *discrete minimization problem*: find $u_h \in V_h$ that minimizes E on V_h. We are interested in the following questions:

1. How are the Ritz method and Galerkin method related?

2. Are these methods solvable?

3. What are their approximation properties?

2.5 Abstract framework

Let $V = H_0^1(\Omega)$. Let

$$a(\varphi, u) = (\varphi, u)_{H_0^1(\Omega)} + (\gamma\varphi, u)_{L^2(\Omega)}.$$

Let

$$L(\varphi) = (\varphi, f)_{L^2(\Omega)}.$$

Proposition 39. *The map $L : V \to \mathbb{R}$ is*

1. *linear: $L(au + v) = aL(u) + L(v)$ for all $a \in \mathbb{R}$ and all $u, v \in V$*
2. *bounded: There is a constant $C > 0$ such that*

   $$|L(\varphi)| \leq C\|\varphi\|_V$$

   for all $\varphi \in V$

Furthermore, the map $a : V \times V \to \mathbb{R}$ is

1. *bilinear: $a(cu + v, w) = ca(u, w) + a(v, w)$ and $a(w, cu + v) = ca(w, u) + a(w, v)$ for all $c \in \mathbb{R}$ and all $u, v, w \in V$*
2. *symmetric: $a(u, v) = a(v, u)$ for all $u, v \in V$*
3. *bounded: there is a constant C such that $|a(u, v)| \leq C\|u\|_V\|v\|_V$ for all $v \in V$*
4. *elliptic: there is a constant $C > 0$ such that $a(u, u) \geq C\|u\|_V$ for all $u \in V$.*

Proof. It is clear that L is linear. Recall that the Cauchy-Schwarz inequality holds for any inner product with its induced norm: $|(u, v)| \leq \|u\|\|v\|$. We also recall that the H^1 norm and the H_0^1 norm are equivalent on $H_0^1(\Omega)$. Thus L is bounded.

It is also clear that a is bilinear and symmetric. Therefore,

$$\begin{aligned}
|a(\varphi, u)| &\leq \|\varphi\|_V\|u\|_V + \|\gamma\varphi\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)} \\
&\leq \|\varphi\|_V\|u\|_V + \|\gamma\|_\infty\|\|\varphi\|_{H^1(\Omega)}\|u\|_{H^1(\Omega)} \\
&\leq (1 + C\|\gamma\|_\infty)\|\varphi\|_V\|u\|_V
\end{aligned}$$

so a is bounded. Finally,

$$a(u, u) = \|u\|_V^2 + (\gamma u, u)_{L^2(\Omega)} \geq \|u\|_V^2$$

so a is elliptic. □

Theorem 40. *The weak problem and the minimization problem are equivalent.*

Proof. Suppose that $u \in H_0^1(\Omega)$ solves the weak problem. Then

$$a(v, u) = L(v)$$

for all $v \in V$. Let $v \in V$ and let $\psi = v - u$. Then

$$\begin{aligned}
E(v) &= \frac{1}{2}a(u + \psi, u + \psi) - L(u + \psi) \\
&= \frac{1}{2}(a(u, u) + 2a(\psi, u) + a(\psi, \psi)) - L(u) - L(\psi) \\
&= \frac{1}{2}a(u, u) - L(u) + a(\psi, \psi) + a(\psi, u) - L(\psi) \\
&= E(u) + a(\psi, \psi) \\
&\geq E(u).
\end{aligned}$$

Since v is arbitrary, u solves the minimization problem.

Now suppose that u solves the minimization problem. Let $\varepsilon \in \mathbb{R}$ and $\varphi \in V$. Let $v = u + \varepsilon\varphi$. Then

$$\begin{aligned}
E(u) &\leq E(v) \\
&= E(u + \varepsilon\varphi) \\
&= \frac{1}{2}(a(u + \varepsilon\varphi, u + \varepsilon\varphi)) - L(u + \varepsilon\varphi) \\
&= E(u) + \varepsilon(a(\varphi, u) - L(\varphi)) + \frac{\epsilon^2}{2}a(\varphi, \varphi).
\end{aligned}$$

Thus

$$0 \leq \varepsilon(a(\varphi, u) - L(\varphi)) + \frac{\epsilon^2}{2}a(\varphi, \varphi)$$

for all $\varepsilon \in \mathbb{R}$ and all $\varphi \in V$. If $\varepsilon > 0$, then

$$0 \leq a(\varphi, u) - L(\varphi) + \frac{\varepsilon}{2}a(\varphi, \varphi).$$

Sending $\varepsilon \to 0$ shows

$$0 \leq a(\varphi, u) - L(\varphi).$$

Now if $\varepsilon < 0$, then

$$0 \geq a(\varphi, u) - L(\varphi) + \frac{\varepsilon}{2}a(\varphi, \varphi).$$

Sending $\varepsilon \to 0$ in this case shows that

$$0 \geq a(\varphi, u) - L(\varphi).$$

Hence u solves the weak problem. □

Corollary 41. *The Galerkin method and the discrete minimization problem are also equivalent.*

Theorem 42. *There is a unique solution to the minimization problem.*

Proof. We make use of all of the properties of L and a in proposition 39. For any $\varphi \in V$,

$$\begin{aligned} E(\varphi) &= \frac{1}{2}a(\varphi, \varphi) - L(\varphi) \\ &\geq \frac{C_0}{2}\|\varphi\|_V^2 - C_1\|\varphi\|_V \end{aligned}$$

Now we recall Young's inequality:

$$ab \leq \frac{1}{2}(\varepsilon a^2 + \frac{1}{\varepsilon}b^2)$$

and we use it with $a = C_1$, $b = \|\varphi\|_V$, and $\varepsilon = \frac{1}{C_0}$:

$$C_1\|\varphi\|_V \leq \frac{C_1^2}{2C_0} + \frac{C_0}{2}\|\varphi\|_V^2.$$

Thus

$$E(\varphi) \geq -\frac{C_1^2}{2C_0} := -C$$

for all $\varphi \in V$. Thus $m_0 := \inf_{v \in V} E(v)$ exists. Recall that for infimums we have that, for all $\varepsilon > 0$, there is $w \in V$ such that

$$E(w) \leq \inf_{v \in V} E(v) + \varepsilon.$$

Thus we can find a minimizing sequence $(v_n) \subset V$ with $E(v_n) \to m_0$. We will show that (v_n) is a Cauchy sequence in V. We have from ellipticity that

$$\|v_m - v_n\|_V \leq Ca(v_m - v_n, v_m - v_n)$$

so it suffices to show that $a(v_m - v_n, v_m - v_n) \to 0$ as $m, n \to \infty$. We have that

$$a(v_m - v_n, v_m - v_n) = a(v_m, v_m) - 2a(v_m, v_n) + a(v_n, v_n)$$

and

$$a(v_m + v_n, v_m + v_n) = a(v_m, v_m) + 2a(v_m, v_n) + a(v_n, v_n).$$

Adding these together gives us

$$a(v_m - v_n, v_m - v_n) + a(v_m + v_n, v_m + v_n) = 2a(v_m, v_m) + 2a(v_n, v_n).$$

This implies that

$$\begin{aligned} a(v_m - v_n, v_m - v_n) &= 2a(v_m, v_m) + 2a(v_n, v_n) - a(v_m + v_n, v_m + v_n) \\ &= 4E(v_m) + 4L(v_m) + 4E(v_n) + 4L(v_n) \\ &\quad - 4a\left(\frac{v_m + v_n}{2}, \frac{v_m + v_n}{2}\right) \pm 8L\left(\frac{v_m + v_n}{2}\right) \\ &= 4E(v_n) + 4E(v_m) - 8E\left(\frac{v_m + v_n}{2}\right) \\ &\leq 4E(v_n) + 4E(v_m) - 8m_0 \to 0 \end{aligned}$$

as $m, n \to \infty$. Hence (v_n) is Cauchy in the complete space V, so there exists u such that $v_n \to u$. Then since E is continuous on V, we have that u solves the minimization problem (and hence also the weak problem).

Now if $v \neq u$, then $\varphi := v - u \neq 0$, so

$$\begin{aligned} 2E(v) &= a(u+\varphi, u+\varphi) - 2L(u+\varphi) \\ &= a(u,u) - 2L(u) + a(\varphi,\varphi) + 2(a(u,\varphi) - L(\varphi)) \\ &= 2E(u) + a(\varphi,\varphi) \\ &> 2E(u) \end{aligned}$$

so v cannot be a minimizer. Hence there is only one minimizer. □