Some Notes on Sobolev Spaces

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1 $L^2(\Omega)$, Inner Products, and Norms

Definition 1. Let Ω be a bounded domain in $\mathbb{R}^d$. The space $L^2(\Omega)$ is the set

$$L^2(\Omega) = \left\{ u : \Omega \to \mathbb{R} : \int_\Omega u(x)^2\, dx < \infty \right\}$$

of all real-valued functions on Ω that are square-integrable. We add two functions $u, v \in L^2(\Omega)$ pointwise: $u + v : \Omega \to \mathbb{R}$ is defined by

$$(u + v)(x) = u(x) + v(x)$$

for all $x \in \Omega$. We also define the scalar multiplication of $u \in L^2(\Omega)$ by $c \in \mathbb{R}$ pointwise: $cu : \Omega \to \mathbb{R}$ is given by

$$(cu)(x) = cu(x)$$

for all $x \in \Omega$. Finally, we define the product of two functions $u, v \in L^2(\Omega)$ pointwise: $uv : \Omega \to \mathbb{R}$ is given by

$$(uv)(x) = u(x)v(x)$$

for all $x \in \Omega$.

Remark 2. With addition and scalar multiplication defined as above, $L^2(\Omega)$ is a vector space. Indeed, given $c \in \mathbb{R}$, and given $u, v \in L^2(\Omega)$, since the identity

$$(a + b)^2 \leq 2a^2 + 2b^2$$

holds for any $a, b \in \mathbb{R}$, we have

$$\int_\Omega (cu(x) + v(x))^2\, dx \leq 2c^2 \int_\Omega u(x)^2\, dx + 2 \int_\Omega v(x)^2\, dx < \infty.$$

Also, since the identity

$$2ab \leq a^2 + b^2$$

holds for any $a, b \in \mathbb{R}$, we have that for any $u, v \in L^2(\Omega)$,

$$\int_\Omega |u(x)v(x)|\, dx \leq \frac{1}{2} \int_\Omega u(x)^2\, dx + \frac{1}{2} \int_\Omega v(x)^2\, dx < \infty.$$

This implies that the quantity

$$\int_\Omega u(x)v(x)\,dx$$

is always a well-defined real number for any $u, v \in L^2(\Omega)$.

Definition 3. Given $u, v \in L^2(\Omega)$, let

$$\langle u, v\rangle_{L^2(\Omega)} = \int_\Omega u(x)v(x)\,dx.$$

Remark 4. Our previous remarks tell us that $\langle u, v\rangle_{L^2(\Omega)} \in \mathbb{R}$ for any $u, v \in L^2(\Omega)$. This defines a map $\langle \cdot, \cdot\rangle_{L^2(\Omega)} : L^2(\Omega) \times L^2(\Omega) \to \mathbb{R}$.

Definition 5. Let V be a vector space. A map $\langle \cdot, \cdot\rangle_V : V \times V \to \mathbb{R}$ is an inner product if

1. (Linearity in the first argument) For any $c \in \mathbb{R}$, any $u_1, u_2, v \in V$,

$$\langle cu_1 + u_2, v\rangle_V = c\langle u_1, v\rangle_V + \langle u_2, v\rangle_V.$$

2. (Symmetry) For any $u, v \in V$,

$$\langle u, v\rangle_V = \langle v, u\rangle_V.$$

3. (Positive-definiteness) For any $u \neq 0$ in V,

$$\langle u, u\rangle_V > 0.$$

Remark 6. Let $\langle \cdot, \cdot\rangle_V$ be an inner product on a vector space V.

1. Linearity in the first argument and symmetry imply that $\langle \cdot, \cdot\rangle_V$ is linear in the second argument: For any $c \in \mathbb{R}$, and $u, v_1, v_2 \in V$,

$$\langle u, cv_1 + v_2\rangle_V = c\langle u, v_1\rangle_V + \langle u, v_2\rangle_V.$$

   Indeed, we have

$$\langle u, cv_1 + v_2\rangle_V = \langle cv_1 + v_2, u\rangle_V = c\langle v_1, u\rangle_V + \langle v_2, u\rangle_V = c\langle u, v_1\rangle_V + \langle u, v_2\rangle_V.$$

   Any map on $V \times V$ that is linear in both arguments is called a bilinear map.

2. The bilinearity of $\langle \cdot, \cdot\rangle$ implies that

$$\langle 0, v\rangle_V = \langle v, 0\rangle_V = 0$$

   for all $v \in V$. Indeed, we have

$$\langle 0, v\rangle_V = \langle 0 + 0, v\rangle_V = \langle 0, v\rangle_V + \langle 0, v\rangle_V.$$

Canceling a $\langle 0, v\rangle_V$ on both sides gives us one of the equalities. A similar proof holds for the other. In particular, we have that

$$\langle 0, 0\rangle_V = 0,$$

and the positive-definiteness property implies that 0 is the only vector in V with this property. Furthermore, we have that

$$\langle u, u\rangle_V \geq 0$$

for all $u \in V$. We refer to this property as being positive semi-definite.

Remark 7. The map $\langle \cdot, \cdot\rangle_{L^2(\Omega)} : L^2(\Omega) \times L^2(\Omega) \to \mathbb{R}$ defined by

$$\langle u, v\rangle_{L^2(\Omega)} = \int_\Omega u(x)v(x)\, dx$$

is an inner product. The linearity and symmetry properties are straightforward from the definitions. For the positive-definiteness property, we would need to introduce some measure theory that is beyond the scope of these notes. For now, we will simply believe this fact. We refer to this inner product as the $L^2(\Omega)$ inner product.

Proposition 8 (The Cauchy-Schwarz Inequality)**.** *Let $\langle \cdot, \cdot\rangle_V$ be an inner product on a vector space V. Then for any $u, v \in V$,*

$$\langle u, v\rangle_V^2 \leq \langle u, u\rangle_V \langle v, v\rangle_V.$$

Proof. If $v = 0$, then the result is trivially true based on our previous remarks. We therefore assume that $v \neq 0$, so that $\langle v, v\rangle_V > 0$. Let $t = \langle u, v\rangle_V / \langle v, v\rangle_V$. Then direct computation shows

$$\begin{aligned} \langle u - tv, u - tv\rangle_V &= \langle u, u\rangle_V - 2t\langle u, v\rangle_V + t^2\langle v, v\rangle_V \\ &= \langle u, u\rangle_V - \frac{\langle u, v\rangle_V^2}{\langle v, v\rangle_V}. \end{aligned}$$

Since $\langle u - tv, u - tv\rangle_V \geq 0$, the inequality follows. □

Remark 9. The inequality above is referred to as the Cauchy-Schwarz inequality. We will write this inequality using slightly different and more common notation later. It is arguably one of the most important and widely used inequalities in all of mathematics.

Definition 10. Let V be a vector space. A map $\|\cdot\|_V : V \to \mathbb{R}$ is a norm on V if

1. (The triangle inequality) For all $u, v \in V$,

$$\|u + v\|_V \leq \|u\|_V + \|v\|_V.$$

2. (Absolute homogeneity) For all $c \in \mathbb{R}$ and all $u \in V$,

$$\|cu\|_V = |c|\|u\|_V.$$

3. (Positive-definiteness) For all $u \neq 0$ in V,

$$\|u\|_V > 0.$$

Remark 11. Let $\|\cdot\|$ be a norm on a vector space V. Then the absolute homogeneity property implies that

$$\|0\|_V = 0.$$

Indeed, we have

$$\|0\|_V = \|2 \cdot 0\|_V = 2\|0\|_V = \|0\|_V + \|0\|_V.$$

Canceling a $\|0\|_V$ on both sides gives us the result. The positive-definiteness property implies that 0 is the only vector in V with this property.

Remark 12. Let $\langle \cdot, \cdot \rangle_V$ be an inner product on a vector space V. Then the map $\|\cdot\|_V : V \to \mathbb{R}$ given by

$$\|u\|_V = \sqrt{\langle u, u \rangle_V}$$

is a norm on V. Indeed, the map is well-defined because inner products are positive semi-definite. The absolute homogeneity property follows from the fact that $\langle \cdot, \cdot \rangle_V$ is bilinear. The fact that $\|\cdot\|_V$ is positive-definite follows from the fact that inner products are positive-definite. Finally, the Cauchy-Schwarz inequality implies that

$$\begin{aligned}
\|u+v\|_V^2 &= \langle u+v, u+v \rangle_V \\
&= \|u\|_V^2 + 2\langle u, v \rangle_V + \|v\|_V^2 \\
&\leq \|u\|_V^2 + 2\|u\|_V\|v\|_V + \|v\|^2 \\
&= (\|u\|_V + \|v\|_V)^2.
\end{aligned}$$

The triangle inequality follows from this.

Definition 13. Let $\langle \cdot, \cdot \rangle_V$ be an inner product on a vector space V. The norm $\|\cdot\|_V$ on V given by

$$\|u\|_V = \sqrt{\langle u, u \rangle_V}$$

is called the induced norm on V.

Remark 14. With a norm $\|\cdot\|_V$ on V induced by an inner product $\langle \cdot, \cdot \rangle_V$, the Cauchy-Schwarz inequality states that

$$|\langle u, v \rangle_V| \leq \|u\|_V \|v\|_V$$

for all $u, v \in V$. This is the more common way to express this inequality.

Remark 15. With the $L^2(\Omega)$ inner product, the induced norm is given by

$$\|u\|_{L^2(\Omega)} = \sqrt{\int_\Omega u(x)^2 \, dx}.$$

We refer to the norm as the $L^2(\Omega)$ norm. Also, the Cauchy-Schwarz inequality in this case reads

$$\left| \int_\Omega u(x)v(x) \, dx \right| \leq \sqrt{\int_\Omega u(x)^2 \, dx} \sqrt{\int_\Omega v(x)^2 \, dx}.$$

We can actually take this a step further: for any $u, v \in L^2(\Omega)$,

$$\left| \int_\Omega u(x)v(x)\,dx \right| \leq \int_\Omega |u(x)||v(x)|\,dx \leq \sqrt{\int_\Omega u(x)^2\,dx}\sqrt{\int_\Omega v(x)^2\,dx},$$

since $u \in L^2(\Omega)$ implies that $|u| \in L^2(\Omega)$ and

$$\|u\|_{L^2(\Omega)} = \||u|\|_{L^2(\Omega)}.$$

2 $H^1(\Omega)$, Weak Derivatives, and Vector Calculus

Definition 16. Let Ω be a bounded domain in $\mathbb{R}^d$. The space $C_0^\infty(\Omega)$ is the set of all functions $\varphi : \Omega \to \mathbb{R}$ that

1. have continuous partial derivatives of all orders, and
2. for which there exists a closed and bounded subset $K_\varphi \subset \Omega$ where $\varphi = 0$ outside of K_φ.

Property 1 is usually called smoothness, and property 2 is usually called being compactly supported. Thus $C_0^\infty(\Omega)$ is the set of all smooth, compactly supported functions on Ω. Some authors use the notation $C_c^\infty(\Omega)$ to denote this set instead.

Remark 17. With addition and scalar multiplication defined pointwise, $C_0^\infty(\Omega)$ is a vector space. Similarly, if multiplication is defined pointwise, then $\varphi\psi \in C_0^\infty(\Omega)$ when $\varphi, \psi \in C_0^\infty(\Omega)$.

Definition 18. Let Ω be a bounded domain in $\mathbb{R}^d$, let $u, v \in L^2(\Omega)$, and $i \in \{1, \ldots, d\}$. We say that v is a weak ith partial derivative of u if for all $\varphi \in C_0^\infty(\Omega)$,

$$\int_\Omega u(x)\partial_i\varphi(x)\,dx = -\int_\Omega v(x)\varphi(x)\,dx.$$

Here, the notation $\partial_i\varphi$ refers to the ith partial derivative of φ in the usual calculus sense. In this context, we usually call functions in $C_0^\infty(\Omega)$ test functions. Also, in this context, we refer to the usual partial derivatives as strong partial derivatives.

Remark 19. If u has a strong ith partial derivative that is square-integrable, then for any smooth, compactly supported test function φ, since $\varphi = 0$ on the boundary of Ω, the usual integration-by-parts formula gives us

$$\int_\Omega u(x)\partial_i\varphi(x)\,dx = -\int_\Omega \partial_i u(x)\varphi(x)\,dx.$$

Therefore, we think of weak partial derivatives as a generalization of strong partial derivatives that still allow us to perform integration-by-parts calculuations when needed. In particular, every strong, square-integrable partial derivative is also a weak partial derivative.

Remark 20. Every function $u \in L^2(\Omega)$ has at most one weak ith partial derivative in $L^2(\Omega)$. Therefore, when it exists, we denote the ith weak partial derivative of $u \in L^2(\Omega)$ by $\partial_i u$. Not every function in $L^2(\Omega)$ has a weak ith partial derivative. We will not prove these facts here, as they require more analysis than we wish to go into.

Remark 21. Most of the usual calculus facts about strong derivatives also carry over to weak derivatives. In particular, if $u, v \in L^2(\Omega)$ have ith weak partial derivatives in $L^2(\Omega)$, then

1. $u + v$ has a weak ith partial derivative in $L^2(\Omega)$ given by
$$\partial_i(u + v) = \partial_i u + \partial_i v,$$
2. for any $c \in \mathbb{R}$, cu has a weak ith partial derivative in $L^2(\Omega)$ given by
$$\partial_i(cu) = c\partial_i u,$$
and

Definition 22. Let Ω be a bounded domain in $\mathbb{R}^d$. Then the space $H^1(\Omega)$ is the set
$$H^1(\Omega) = \{u \in L^2(\Omega) : \partial_i u \in L^2(\Omega) \text{ for each } i = 1, \dots, d\}$$
of all functions $u \in L^2(\Omega)$ that have weak ith partial derivatives in $L^2(\Omega)$ for each $i \in \{1, \dots, d\}$. Sometimes we will simply call the weak partial derivatives as partial derivatives.

Remark 23. With addition and scalar multiplication defined pointwise, $H^1(\Omega)$ is a vector space. Since it is a subspace of $L^2(\Omega)$, the $L^2(\Omega)$ inner product / induced norm also gives us an inner product / norm on $H^1(\Omega)$. However, we can also define a new inner product / norm on $H^1(\Omega)$ that encodes information about the weak derivatives of a function in $H^1(\Omega)$.

Definition 24. Given $u, v \in H^1(\Omega)$, let
$$\langle u, v \rangle_{H^1(\Omega)} = \int_\Omega u(x)v(x) + \sum_{i=1}^d \partial_i u(x)\partial_i v(x)\, dx.$$

Remark 25. A few remarks about the above definition are in order:

1. Since u, v, and its partial derivatives are in $L^2(\Omega)$, the quantity $\langle u, v \rangle_{H^1(\Omega)}$ is a well-defined real number.
2. Recall from vector calculus that the gradient of a differentiable scalar-valued function $u : \Omega \to \mathbb{R}$ is the function $\nabla u : \Omega \to \mathbb{R}^d$ given by
$$\nabla u = (\partial_1 u, \dots, \partial_d u).$$
We can extend this notation to $H^1(\Omega)$ by defining the (weak) gradient of $u \in H^1(\Omega)$ to be the map $\nabla u : \Omega \to \mathbb{R}^d$ given by the same formula. If we also recall that the dot product of two vectors $x, y \in \mathbb{R}^d$ is given by
$$x \cdot y = \sum_{i=1}^d x_i y_i,$$
then we have that
$$\langle u, v \rangle_{H^1(\Omega)} = \int_\Omega u(x)v(x) + \nabla u(x) \cdot \nabla v(x)\, dx.$$

3. We can also write

$$\langle u, v\rangle_{H^1(\Omega)} = \langle u, v\rangle_{L^2(\Omega)} + \sum_{i=1}^{d} \langle \partial_i u, \partial_i v\rangle_{L^2(\Omega)} = \langle u, v\rangle_{L^2(\Omega)} + \langle \nabla u, \nabla v\rangle_{L^2(\Omega)},$$

   where the notation

$$\langle \nabla u, \nabla v\rangle_{L^2(\Omega)} := \sum_{i=1}^{d} \langle \partial_i u, \partial_i v\rangle_{L^2(\Omega)}.$$

4. The map $\langle \cdot, \cdot\rangle_{H^1(\Omega)} : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ defined by the formula above is an inner product on $H^1(\Omega)$. This is most easily proven from item 3. We call this inner product the $H^1(\Omega)$ inner product.

5. The induced norm from the $H^1(\Omega)$ inner product is given by

$$\begin{aligned} \|u\|_{H^1(\Omega)} &= \sqrt{\int_\Omega u(x)^2 + \sum_{i=1}^{d} (\partial_i u(x))^2 \, dx} \\ &= \sqrt{\int_\Omega u(x)^2 + |\nabla u(x)|^2 \, dx} \\ &= \sqrt{\|u\|_{L^2(\Omega)}^2 + \sum_{i=1}^{d} \|\partial_i u\|_{L^2(\Omega)}^2} \\ &= \sqrt{\|u\|_{L^2(\Omega)}^2 + \|\nabla u\|_{L^2(\Omega)}^2}, \end{aligned}$$

   where we recall that the usual Euclidean norm of a vector $x \in \mathbb{R}^d$ is given by

$$|x| = \sqrt{\sum_{i=1}^{d} x_i^2},$$

   and the notation

$$\|\nabla u\|_{L^2(\Omega)} := \||\nabla u|\|_{L^2(\Omega)}.$$

6. From item 5, its immediate that

$$\|u\|_{L^2(\Omega)} \leq \|u\|_{H^1(\Omega)}$$

   for every $u \in H^1(\Omega)$.

3 $H_0^1(\Omega)$ and the Poincaré Inequality

Definition 26. Let Ω be a bounded domain in $\mathbb{R}^d$ with boundary $\partial\Omega$. The space $H_0^1(\Omega)$ is the set

$$H_0^1(\Omega) = \{u \in H^1(\Omega) : u = 0 \text{ on } \partial\Omega\}.$$

Remark 27. With addition and scalar multiplication defined pointwise, $H^1(\Omega)$ is a vector space. Since it is a subspace of $H^1(\Omega)$, both the $H^1(\Omega)$ and the $L^2(\Omega)$ inner products / norms give inner products / norms on $H_0^1(\Omega)$. However, we can use the fact that functions in $H^1(\Omega)$ vanish on the boundary to give us a more convenient norm / inner product to work with.

Theorem 28 (Poincaré Inequality)**.** *Let Ω be a bounded domain in $\mathbb{R}^d$. There exists a constant $C_P > 0$ such that for all $u \in H_0^1(\Omega)$,*

$$\|u\|_{L^2(\Omega)} \leq C_P \|\nabla u\|_{L^2(\Omega)}.$$

Proof. We will not prove this here, as it involves some analysis that is beyond the scope of these notes. $\square$

Remark 29. The inequality above is known as a Poincaré inequality. It relates the norm of a function with the norm of its derivative. This inequality is usually expressed as

$$\int_\Omega u(x)^2 \, dx \leq C \int_\Omega |\nabla u(x)|^2 \, dx.$$

Setting $C = C_P^2$ and taking square roots gives us our version of the inequality. We stress that this result holds on $H_0^1(\Omega)$, but not on $H^1(\Omega)$. As a counterexample, take $u(x) = 1$ for $x \in \Omega$. Then $u \in H^1(\Omega)$, $\|u\|_{L^2(\Omega)}$ is the volume of Ω, which is assumed to be strictly positive, and $\|\nabla u\|_{L^2(\Omega)} = 0$.

Definition 30. Let $\|\cdot\|_1, \|\cdot\|_2$ be two norms on a vector space V. Then the two norms are equivalent if there are constants $c, C > 0$ such that

$$c\|u\|_1 \leq \|u\|_2 \leq C\|u\|_1$$

for all $u \in V$.

Remark 31. The previous definition can be intuitively thought of as saying that two norms on a vector space are essentially the same. For instance, if two norms are equivalent, then any inequalities $\|u\|_1 \leq \|v\|_1$ in one of the norms also imply $\|u\|_2 \leq C\|v\|_2$ in the other norm for some constant C that is independent of u and v. Also, any sequences that converge in one norm must also converge in the other norm.

Corollary 32. *On $H_0^1(\Omega)$,*

1. *the map $\langle \cdot, \cdot \rangle_{H_0^1(\Omega)} : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ given by*

$$\langle u, v \rangle_{H_0^1(\Omega)} = \langle \nabla u, \nabla v \rangle_{L^2(\Omega)}$$

   is an inner product,

2. *for the induced norm $\|\cdot\|_{H_0^1(\Omega)} = \sqrt{\langle \nabla \cdot, \nabla \cdot \rangle_{L^2(\Omega)}}$, there is a constant $C_P > 0$ such that for any $u \in H_0^1(\Omega)$,*

$$\|u\|_{L^2(\Omega)} \leq C_P \|u\|_{H_0^1(\Omega)}.$$

3. the induced norm $\|\cdot\|_{H_0^1(\Omega)}$ is equivalent to the $H^1(\Omega)$ norm,

Proof. The map $\langle\cdot,\cdot\rangle_{H_0^1(\Omega)}$ is well-defined, symmetric, and linear in the first argument from our previous work. To see that it is positive-definite, let $u \in H_0^1(\Omega)$ be such that $u \neq 0$. Then the Poincaré inequality implies

$$0 < \|u\|_{L^2(\Omega)}^2 \leq C_P^2 \langle u, u\rangle_{H_0^1(\Omega)}.$$

Thus $\langle\cdot,\cdot\rangle_{H_0^1(\Omega)}$ is an inner product.

The Poincaré inequality and the definition of the induced norm also shows that

$$\|u\|_{L^2(\Omega)} \leq C_P \|u\|_{H_0^1(\Omega)}.$$

We have immediately from the definitions that

$$\|u\|_{H_0^1(\Omega)} \leq \|u\|_{H^1(\Omega)}.$$

On the other hand, using the definitions of the different norms plus our observations above gives us

$$\|u\|_{H^1(\Omega)}^2 = \|u\|_{L^2(\Omega)}^2 + \|u\|_{H_0^1(\Omega)}^2 \leq (C_P + 1)\|u\|_{H_0^1(\Omega)}^2.$$

Taking square roots shows us that

$$\|u\|_{H_0^1(\Omega)} \leq \|u\|_{H^1(\Omega)} \leq \sqrt{C_P + 1}\|u\|_{H_0^1(\Omega)}.$$

Thus the two norms are equivalent on $H_0^1(\Omega)$. $\square$

Remark 33. Since this proof relies on the Poincaré inequality, we stress that the $H_0^1(\Omega)$ inner product / norm, despite being well-defined for functions in $H^1(\Omega)$, is not necessarily an inner product / norm on $H^1(\Omega)$. The only condition that is violated is the positive-definite condition. As a counterexample, take $u(x) = 1$ for $x \in \Omega$. Then $u \neq 0$, but $\nabla u = 0$, so $\langle u, u\rangle_{H_0^1(\Omega)} = \|u\|_{H_0^1(\Omega)} = 0$.