Structure-Preserving Finite Element Schemes for the Euler-Poisson Equations

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Introduction

The Euler-Poisson equations are a coupled system where the compressible Euler equations of gas dynamics are coupled to a scalar potential that satisfies Poisson's equation. The system has applications in plasma physics, semiconductor device modeling, and vacuum electronics, where it is often used to model an electron fluid that is subject to electrostatic forces.

Our goal is to develop structure-preserving finite element schemes to numerically solve the Euler-Poisson equations. By a structure-preserving scheme, we mean that we wish for our numerical schemes to preserve discrete versions of certain properties that are enjoyed by the system at the PDE level (e.g. conservation of energy, invariant-domain properties, etc.). The schemes that we present here accomplish such things. We will discuss some of the algorithmic details and key structure-preserving properties of the schemes.

The work done here is a first step towards developing robust numerical methods for the full Euler-Maxwell system, which accounts for magnetic field effects that are neglected in the Euler-Poisson system.

Model

The **Euler-Poisson equations** are

$$\partial_t \rho + \operatorname{div} \mathbf{m} = 0, \tag{1a}$$

$$\partial_t \mathbf{m} + \operatorname{div}\left(\frac{1}{\rho}\mathbf{m}\mathbf{m}^{\mathsf{T}} + \mathbf{I}p\right) = -\rho\nabla\varphi, \tag{1b}$$

$$\partial_t \mathcal{E} + \operatorname{div}\left(\frac{\mathbf{m}}{\rho}(\mathcal{E}+p)\right) = -\nabla\varphi\cdot\mathbf{m}, \tag{1c}$$

$$-\Delta\varphi = \alpha\rho. \tag{1d}$$

Here, $\rho(t,\mathbf{x}) \in (0,\infty)$ is the **mass density**, $\mathbf{m}(t,\mathbf{x}) \in \mathbb{R}^d$ is the **momentum density**, $\mathcal{E}(t,\mathbf{x}) \in (0,\infty)$ is the **total energy density**, $\varphi(t,\mathbf{x}) \in \mathbb{R}$ is the **scalar potential**, $p(t,\mathbf{x}) \in \mathbb{R}$ is the **thermodynamic pressure**, $\alpha \in \mathbb{R}$ is the **coupling constant**, and $\mathbf{I} \in \mathbb{R}^{d\times d}$ is the identity matrix. The domain of this system is denoted by $[0,\infty)\times\Omega$, where $\Omega \subset \mathbb{R}^d$ is a bounded domain with boundary $\partial\Omega$.

We close the system by assuming that the pressure is given by the following **equation of state** that comes from the ideal gas law:

$$p = (\gamma-1)\left(\mathcal{E} - \frac{1}{2\rho}|\mathbf{m}|_{\ell^2}^2\right)$$

with $\gamma = 5/3$.

PDE Energy Balance

Formally, system (1) has the following **energy balance**:

$$\frac{d}{dt}\int_\Omega \mathcal{E} + \frac{1}{2\alpha}|\nabla\varphi|_{\ell^2}^2\,dx + \int_{\partial\Omega}\left\{\frac{\mathbf{m}}{\rho}(\mathcal{E}+p) + \varphi\left(\mathbf{m} - \frac{1}{\alpha}\partial_t\nabla\varphi\right)\right\}\cdot\mathbf{n}\,ds = 0. \tag{2}$$

We wish to preserve this at the discrete level. To avoid technicalities, we hereby assume that we have prescribed appropriate boundary conditions to (1) to ensure that the boundary term above vanishes.

Operator Splitting and Finite Element Spaces

We can take the time derivative of equation (1d) and use (1a) to rewrite (1d) as an **evolution equation**

$$\partial_t\Delta\varphi = \alpha \operatorname{div}\mathbf{m}.$$

Replacing (1d) with this equation allows us to write system (1) in the form of an **operator splitting**:

$$\begin{aligned}\partial_t\mathbf{u} &= \operatorname{div}\mathbf{f}(\mathbf{u}) + \mathbf{g}(\mathbf{u},\varphi),\\ \partial_t\Delta\varphi &= 0 \qquad + \alpha\operatorname{div}\mathbf{m},\end{aligned}$$

where $\mathbf{u} = (\rho, \mathbf{m}, \mathcal{E})^{\mathsf{T}}$,

$$\mathbf{f}(\mathbf{u}) = \begin{bmatrix}\mathbf{m}^{\mathsf{T}}\\ \frac{1}{\rho}\mathbf{m}\mathbf{m}^{\mathsf{T}} + \mathbf{I}p\\ \frac{\mathbf{m}^{\mathsf{T}}}{\rho}(\mathcal{E}+p)\end{bmatrix}, \quad \mathbf{g}(\mathbf{u},\varphi) = \begin{bmatrix}0\\ -\rho\nabla\varphi\\ -\nabla\varphi\cdot\mathbf{m}\end{bmatrix}$$

We call the first stage in this split the **hyperbolic system** and the second stage the **source-dominated system**. We will proceed to **discretize each system separately** to get partial updates and then combine the two discretizations for a full update.

For simplicity of exposition, we assume that the spatial domain Ω can be meshed by an **affine family of triangular** (if $d = 2$) **or tetrahedral** (if $d = 3$) **meshes** $\{\mathcal{T}_h\}_{h>0}$.

For a given mesh $\mathcal{T}_h$, we assume that each physical element $K \in \mathcal{T}_h$ is mapped by an **affine reference transformation** $\mathbf{T}_K : \hat{K} \to K$ from the **reference element** $\hat{K}$.

We now define the following **finite element spaces**:

$$\begin{aligned}\mathbb{V}_h &= \{z_h \in L^2(\Omega) : z_h \circ \mathbf{T}_K \in \mathbb{P}^1(\hat{K}) \;\forall\, K \in \mathcal{T}_h\},\\ \mathbb{H}_h &= \{\omega_h \in \mathcal{C}^0(\Omega) : \omega_h \circ \mathbf{T}_K \in \mathbb{P}^1(\hat{K}) \;\forall\, K \in \mathcal{T}_h\},\end{aligned}$$

where $\mathbb{P}^1(\hat{K})$ is the space of **degree 1 polynomials** on $\hat{K}$.

Now we let $\{\phi_i\}_i$ and $\{\chi_i\}_i$ be nodal bases of $\mathbb{V}_h$ and $\mathbb{H}_h$ respectively. With respect to these bases, our approximations to $\mathbf{u}, \varphi$ at time t_n are denoted

$$\mathbf{u}_h^n = \sum_i \mathbf{U}_i^n\phi_i, \qquad \varphi_h^n = \sum_i \varphi_i^n\chi_i$$

with $\mathbf{U}_i^n = (\varrho_i^n, \mathbf{M}_i^n, \mathcal{E}_i^n)^{\mathsf{T}}$.

To obtain some discrete energy balances, we also introduce the **lumped inner product** of $\mathbf{f}, \mathbf{g} \in [\mathcal{C}^0(\mathcal{T}_h)]^d$ to be

$$\langle f, g\rangle = \sum_{K\in\mathcal{T}_h}\sum_k f(\mathbf{x}_{K,k})g(\mathbf{x}_{K,k})w_{K,k},$$

where $\mathbf{x}_{K,k} = \mathbf{T}_K(\hat{x}_k)$, $w_{K,k} = \int_K \phi_{K,k}\,dx$, and $\phi_{K,k} = \hat{\phi}_k \circ \mathbf{T}_K^{-1}$.

Hyperbolic Update

For the hyperbolic subsystem, we use a discretization technique that was first developed in [2] which relies on **graph viscosity** and **convex limiting**. This gives us the **hyperbolic update** $(\rho_h^n, \mathbf{m}_h^n, \mathcal{E}_h^n) \to (\rho_h^{n+1}, \mathbf{m}_h^{n+1}, \mathcal{E}_h^{n+1})$. The **hyperbolic update** is **robust**, **high order**, and **maintains the right PDE structural properties** (conservation, invariant domains, entropy inequalities) that are important to the theory of systems of hyperbolic conservation laws. Furthermore, only a **hyperbolic CFL condition** on the time-step size τ_n is needed to guarantee these properties.

Source Update

We first observe that the **source-dominated** system has the following formal **energy balance**:

$$\frac{d}{dt}\int_\Omega \frac{1}{2\rho}|\mathbf{m}|_{\ell^2}^2 + \frac{1}{2\alpha}|\nabla\varphi|_{\ell^2}^2\,dx + \int_{\partial\Omega}\left\{\varphi\left(\mathbf{m} - \frac{1}{\alpha}\partial_t\nabla\varphi\right)\right\}\cdot\mathbf{n}\,ds = 0. \tag{3}$$

We assume that our boundary conditions are such that the boundary term above is 0. **We wish to preserve this energy balance at the discrete level**.

A fully discrete version of the source-dominated system with a **Crank-Nicolson discretization in time** is as follows. Given ρ_h^n, $\mathbf{v}_h^n$, and φ_h^n, at time t_n, find $\mathbf{v}_h^{n+1}$ and φ_h^{n+1} at time $t_{n+1} = t_n + \tau_n$ that satisfy

$$\begin{aligned}\langle\rho_h^n\mathbf{v}_h^{n+1}, \mathbf{z}_h\rangle = \langle\rho_h^n\mathbf{v}_h^n, \mathbf{z}_h\rangle - \frac{\tau_n}{2}\langle\rho_h^n\{\nabla\varphi_h^{n+1}\\ + \nabla\varphi_h^n\}, \mathbf{z}_h\rangle,\end{aligned} \tag{4a}$$

$$a_{\tau_n}^+(\varphi_h^{n+1}, \omega_h) = a_{\tau_n}^-(\varphi_h^n, \omega_h) + \tau_n\alpha\langle\rho_h^n\mathbf{v}_h^n, \nabla\omega_h\rangle, \tag{4b}$$

for all $\omega_h \in \mathbb{H}_h$ and all $\mathbf{z}_h \in [\mathbb{V}_h]^d$, where

$$a_{\tau_n}^\pm(\varphi,\omega) = (\nabla\varphi, \nabla\omega) \pm \frac{\tau_n^2\alpha}{4}(\rho_h^n\nabla\varphi, \nabla\omega). \tag{5}$$

After getting $\mathbf{v}_h^{n+1} = \sum_i \mathbf{V}_i^{n+1}\phi_i$, we get $\mathbf{m}_h^{n+1} = \sum_i \mathbf{M}_i^{n+1}\phi_i$ by setting $\mathbf{M}_i^{n+1} = \varrho_i^n\mathbf{V}_i^{n+1}$.

This system is **well-posed**, **efficient** to solve, and the time-step τ_n is only subjected to the **hyperbolic CFL condition**.

Most importantly, system (4) has the following **discrete energy balance**:

$$\begin{aligned}\frac{1}{2}\sum_i \varrho_i^n|\mathbf{V}_i^{n+1}|_{\ell^2}^2 + \frac{1}{2\alpha}\|\nabla\varphi_h^{n+1}\|_{\mathbf{L}^2(\Omega)}^2\\ = \frac{1}{2}\sum_i \varrho_i^n|\mathbf{V}_i^n|_{\ell^2}^2 + \frac{1}{2\alpha}\|\nabla\varphi_h^n\|_{\mathbf{L}^2(\Omega)}^2.\end{aligned}$$

This is exactly the discrete version of (3) (minus the boundary term) that we want.

To update the energy, we first observe that the source-dominated system satisfies

$$\partial_t\left(\mathcal{E} - \frac{1}{2\rho}|\mathbf{m}|_{\ell^2}^2\right) = 0.$$

Therefore, after getting $\mathbf{m}_h^{n+1}$, we **update the energy** $\mathcal{E}_h^{n+1} = \sum_i \mathcal{E}_i^{n+1}\phi_i$ by

$$\mathcal{E}_i^{n+1} = \mathcal{E}_i^n + \frac{1}{2\varrho_i^{n+1}}\left(|\mathbf{M}_i^{n+1}|_{\ell^2}^2 - |\mathbf{M}_i^n|_{\ell^2}^2\right). \tag{6}$$

The source update procedure given by (4) and (6) is **second-order**, **robust**, and **structure-preserving**.

Full Update Procedure

We combine the hyperbolic update with the source update to get a **complete update** as follows:

1. Given $\mathbf{u}_h^n = (\rho_h^n, \mathbf{m}_h^n, \mathcal{E}_h^n)^{\mathsf{T}}$ and φ_h^n at time t_n, compute the time-step size τ_n subjected to a hyperbolic CFL condition and update
$$\mathbf{u}_h^n \to \mathbf{u}_h^{n+1,1} = (\rho_h^{n+1,1}, \mathbf{m}_h^{n+1,1}, \mathcal{E}_h^{n+1,1})^{\mathsf{T}}$$
at time $t_{n+1} = t_n + \tau_n$ via the hyperbolic update.
2. Feed the partial hyperbolic update $\mathbf{u}_h^{n+1,1}$ and φ_h^n into the source update scheme to get the full update
$$(\rho_h^{n+1,1}, \mathbf{m}_h^{n+1,1}, \mathcal{E}_h^{n+1,1}, \varphi_h^n) \to (\rho_h^{n+1}, \mathbf{m}_h^{n+1}, \mathcal{E}_h^{n+1}, \varphi_h^{n+1})$$

The update algorithm given above is a **first-order - Yanenko operator splitting** algorithm and was presented for simplicity. However, one can modify the algorithm to get a **second-order Strang operator splitting** algorithm that still maintains the stability and structure-preserving properties of the first-order split, but now is also **high order**.

The schemes presented here use affine triangular meshes and $\mathbb{P}^1$ elements for simplicity of exposition, but they can be extended to asymptotically affine families of quadrilateral meshes with $\mathbb{Q}^1$ elements [3]. Only minimal changes to the algorithm are necessary.

This full update procedure maintains the following **discrete energy balance**:

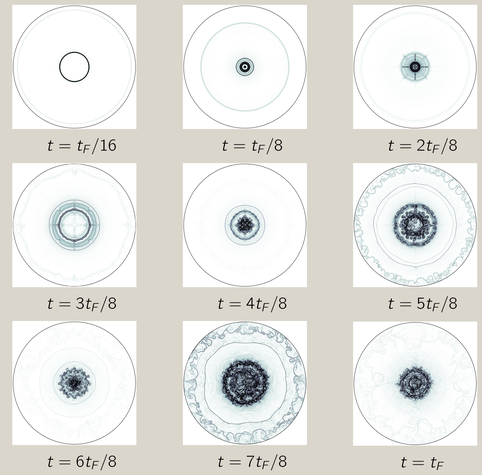
$$\sum_i m_i\mathcal{E}_i^{n+1} + \frac{1}{2\alpha}\|\nabla\varphi_h^{n+1}\|_{L^2(\Omega)}^2 = \sum_i m_i\mathcal{E}_i^n + \frac{1}{2\alpha}\|\nabla\varphi_h^n\|_{L^2(\Omega)}^2,$$

which is exactly a discrete counterpart to the PDE energy balance (2) (minus the boundary term) that we were hoping to preserve. Therefore, under a hyperbolic CFL condition, the scheme above is **stable** and **structure-preserving**.

Numerical Illustration: Electrostatic Implosion

To numerically test our method, we consider an **electrostatic implosion configuration** in a **circular domain** $\Omega = \{\mathbf{x} \in \mathbb{R}^2 : |x|_{\ell^2} \le r_3\}$ of radius $r_3 = 16$, with boundary conditions $\mathbf{m}\cdot\mathbf{n} = 0$ and $\varphi = 0$ on $\partial\Omega$ and coupling constant $\alpha = 10^3$. The initial state is as follows: the density ρ is initially $\rho(\mathbf{x}) = 10001$ if $r_1 = 4 \le |x|_{\ell^2} \le 6 = r_2$ and $\rho(\mathbf{x}) = 1$ otherwise, the velocity $\mathbf{v}_0 = \mathbf{0}$ uniformly, and the pressure $p_0 = 10^{-4}$ uniformly. The final time is set to $t_F = (3/64)t_P$ where $t_P = 2\pi/\omega_p$ and $\omega_p = \sqrt{10^3}$.

The geometric setup is similar to considering a configuration with two concentric cylindrical electrodes, with the outer electrode grounded and the inner electrode having a very high positive voltage pulling the electron gas inwards. As the gas is accelerated towards the center, a cylindrical outer region with very low density and low pressure is left behind. Such a configuration is known to be **hydrodynamically highly unstable**, so this is an excellent test to see if our scheme can work in the **shock hydrodynamics regime**. The computation was carried out in **deal.II** [1] with approximately 1M quadrilaterals. The figure below shows various snapshots in time of the schlieren plot of the density, which shows that our scheme appears to accurately capture the dynamics of the system.



Outlook

The schemes presented here give us a **robust**, **high order**, **structure-preserving** numerical discretization of the Euler-Poisson equations that work for triangular, quadrilateral, or hexagonal families of meshes, both affine and asymptotically affine. The structure-preserving properties of these schemes are provably verified, and their accuracy and efficiency are demonstrated through various numerical numerical tests which can be found in [3].

The Euler-Poisson equations themselves can be thought of as the **electrostatic limit** of the larger **Euler-Maxwell equations**:

$$\begin{aligned}\partial_t\rho + \operatorname{div}\mathbf{m} &= 0,\\ \partial_t\mathbf{m} + \operatorname{div}\left(\frac{1}{\rho}\mathbf{m}\mathbf{m}^{\mathsf{T}} + \mathbf{I}p\right) &= -\rho(\nabla\varphi + \partial_t\mathbf{A})\\ &\quad + \mathbf{m}\times\operatorname{curl}\mathbf{A},\\ \partial_t\mathcal{E} + \operatorname{div}\left(\frac{\mathbf{m}}{\rho}(\mathcal{E}+p)\right) &= -(\nabla\varphi + \partial_t\mathbf{A})\cdot\mathbf{m},\\ -\Delta\varphi - \partial_t\operatorname{div}\mathbf{A} &= \alpha\rho,\\ \frac{1}{c^2}\partial_t\nabla\varphi + \frac{1}{c^2}\partial_t^2\mathbf{A} + \operatorname{curl}^2\mathbf{A} &= \frac{\alpha}{c^2}\mathbf{m}\end{aligned}$$

Here, φ is the **electric potential** and $\mathbf{A}$ is the **magnetic potential**. **Our goal in a future publication is to extend our methods to the Euler-Maxwell equations.**

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